ON THE EXTENSION OF THE BESSEL FUNCTIONS OF THE FIRST KIND WITH APPLICATIONS TO SOME MATHEMATICAL INTEGRALS AND TRANSFORMS

BY

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To my parents, wife, kids, brothers and sisters

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I am in full indebtedness to Allah for the whole provided so as the completeness of this work is just one out of much. I really ask him to help me and support me and get me to his mercy in all of my breathing moments.

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DISSERTATION ABSTRACT

Name: AbdulKhaleg Ali Al-Baiyat

Title of study: On the Extension of the Bessel Functions of the First Kind with Applications to some Mathematical Integrals and Transforms

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Bessel Functions that have been known in the literature and are applied in many areas in engineering fields will be extended here. They will be defined through the extension of the recently defined extended confluent hypergeometric functions. We will provide some insights into the original function which is directly related to the Bessel function and the modified Bessel function of the first kind. As such, its extension provides an extension of the two Bessel functions of the first kind as well. In particular, it provides an integral representation which *yields an integral formula relating the standard Bessel function of any order to the Bessel function of order 1.* Indeed, as the extension is carried out through the extension of the beta function, some important results have been derived for this function too. In particular, it is shown that the difference between the function with first variable shifted by any integer $n \geq 1$ and that of the function with the first variable shifted by one is the same as the corresponding difference for the second variable.

As the application part is the most important part of any extension, we have applied our extension to develop the two known integral identities (Lipschitz and Hankel's) into closed mathematical forms. The Laplace transform has also been obtained for the extended Bessel function of the first kind of order zero. *Interestingly, the closed form of the transform is expressed in terms of the extended hypergeometric function, which has also been recently introduced*. Studies of the extended Bessel function of the first kind as the asymptotic behavior and the Mellin-Barnes integral have also been covered in this work. On the continuation of the application part, we prove that the extended Bessel function of the first kind

satisfies a non homogeneous second order differential equation. The classical Bessel differential equation satisfied by the standard Bessel function is deduced as a special case of this non homogeneous differential equation.

ملخص بحث

درجة الدكتوراه في الفلسفة

الاســـــــــــم: عبدالخالق علي حسن البيات

عنـــوان الرســـالة: تعميم دوال بسل من النوع الأول مع بعض التطبيقات في التكاملات والتحويلات الرياضية

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سوف نقوم في هذه الدراسة بتعميم دوال بسل ذات التطبيقات الكثيرة في المجالات الهندسية. وسوف يكون ذلك من خلال تعميم دالة الكنفلونت هايبرجيومترك. حيث إننا سنقوم ببعض الدراسة للدالة الأصلية، والتي يمكن ربطها مباشرة مع دوال بسل الأصلية والمغيّرة من النوع الأول. ولذلك فإن تعميمها (دالة الكنفلونت هايبرجيومترك) سوف يقود لتعميم كل من الدالتين الأخرتين.

كما أن هذه الدراسة سوف تعطينا تمثيلا (قانونا) في صورة التكامل *يربط بين* جميع *دوال بسل من النوع الأول مع دالة بسل من النوع الأول من الدرجة الأولى*. وأما بالنسبة لطريقة التعميم، فإن ذلك سيكون من خلال تعميم دالة البيتا ذات المتغيرين (س،ص)، والتي سوف نقوم بتقديم نتائج مهمة حولها في هذه الدراسة. وعلى وجه الخصوص، فإننا سنقدم نتيجة مهمة توضّح أن *الفرق بين أي دالتين بيتا من خلال الإزاحة لمتغيرها الأول (س) بأي مقدار (ن $\leq$ 1، حيث ن عدد طبيعي) يساوي الفرق بين أي دالتين بيتا من خلال الإزاحة لمتغيرها الثاني (ص) بنفس مقدار الإزاحة*.

وأما بالنسبة لتطبيقات هذه الدالة (دالة بسل المعممة)، فقد قمنا بتعميم تكامُلي (ليبشيتز وهنكل العام) وإيجاد النتائج في صورة رياضية محكمة. كما قمنا بإيجاد تحويل (لابلاس) لدالة بسل المعممة من النوع الأول من الدرجة الصفر. ومما يدعو للاهتمام هنا هو أنه لم يكن بالإمكان كتابة هذا التحويل بصورة محكمة إلا من خلال دالة الهيبرجيومترك المعممة الحديثة التعريف. كما قمنا أيضا في هذا العمل بدراسة السلوك التقاربي للدالة، وإيجاد تكامل (ميلن – بيرنز).

وفي نهاية البحث قمنا بإيجاد معادلة تفاضلية غير متجانسة تعد بمثابة التعميم لمعادلة بسل التفاضلية المتجانسة المعروفة. وقد أثبتنا أن دالة بسل المعممة من النوع الأول تحقق هذه المعادلة التفاضلية التي أوجدناها.

Introduction

The area of *special functions* has become of more importance to engineering problems and the demand of that has expanded a lot [1, 2, 16, 32]. Closed form solutions to differential equations resulting from applications and the simplification of complicated series forms or results are one of the most important points to be considered and studied. So, from this we expect to have so many of such special functions. For example, we have the gamma function, the beta function, the zeta function,...etc. Of course, the Bessel function which is the core of our study is one of the important examples of such special functions.

At this stage, one would explore the most general special function from which all other special functions could be expressed. That is to find a general function from which all other functions would be derived as special cases of it. So, the Gauss hypergeometric function ${}_2F_1(a,b;c;z)$ appeared in the literature and its properties have been studied extensively. Later, ${}_pF_q(a_1,a_2,...,a_p;b_1,b_2,...,b_q;z)$, the generalized Gauss hypergeometric function was introduced and again its properties have been covered extensively. For the details we refer to [2, 33, 34, 36].

Recently, extension of such special functions has appeared in the research areas. Of course, so many different ways could be far possible extensions but which one would be the best choice, is a raised question. Among all discussions [29] a good extension would be the one that preserves most properties of the original function and that would give elegant new relations on the newly defined extended one.

Some extensions to the gamma and beta functions have been made [3]. However, Chaudhry et. al. have suggested new extensions of some of the important special functions that we are going to follow in this work. They made the extension of the gamma function, the incomplete gamma function, the beta function, the incomplete beta function, the Gauss hypergeometric function and others. In addition, they have made extensive study on their newly introduced functions. For more details, we refer to [9, 10, 11, 12].

Here in our study, we pick up one well known special function which is the Bessel function and we follow Chaudhry approach in extending it and study some of its properties. So, the plan of this research work would be as follows.

In chapter 1, we provide some background on the gamma function, the beta function and the Bessel function that is available in the literature. Then, in chapter 2, some basic and important results on the extended complete and incomplete beta function are discussed.

In chapter 3, and after giving some new results on both the confluent hypergeometric function and the Bessel function, we define and extended confluent hypergeometric and the extended Bessel functions. In that we see *how both the standard and extended Bessel functions are expressible in terms of the standard Bessel function of the first kind of order one*. We also define the incomplete version of both of them and their relations with the complete one. Then, some graphical representations of the extended Bessel function are presented on a separate section.

In chapter 4, we apply our results in finding the Lipschitz and Hankel's integrals for the extended Bessel function. For that, we define the generalized extended hypergeometric function. Then, we study the Laplace transform and the Mellin-Barnes contour integral.

Finally, in the last chapter, we develop some new recurrence relations for the extended Bessel functions. Such relations would be used in generalizing the Bessel differential equation that is going to be presented together with its general solution.

Chapter 1

Preliminaries

1.1 The Beta Function and its Relation with the Gamma Function

As the beta function and the gamma function are correlated strongly with each other, it is of importance to give some insight on both of them. It is also of importance to give some glance on how both of them have been extended and to what extent such extension have preserved their relationship.

Actually, there are different representations for the gamma and the beta functions which imply simplifications in the derivation of certain properties as we will see. So, we start with the gamma function and with its well known integral representation, along with a discussion of the beta function. Then, we see how these two functions are correlated with each other.

It is to be remarked here that all the properties that are presented in this chapter are all available in the literature. However, we presented them here with their proofs (which are of course not ours) just for the sake of having a complete view of what we are doing on this topic and how it is going to be developed later on. Our contributions will come on the chapters that follow.

Definition 1.1.1. *The gamma function is defined by*

$$\Gamma(\alpha) := \int_0^\infty t^{\alpha-1}e^{-t}\,dt \qquad (Re(\alpha) > 0). \tag{1.1}$$

Definition 1.1.2. *The beta function is defined by [16, p.18 (1)]*

$$\beta(u,v) := \int_0^1 t^{u-1}(1-t)^{v-1}dt \qquad (Re(u) > 0\,,\; Re(v) > 0). \tag{1.2}$$

This beta function, which is also called Eulerian integral of the first kind, was introduced by Euler (1707-1783) [2]. Although he was the first who concerned about the numbers

$$n! = \int_0^\infty e^{-t}t^n dt,$$

with nonintegral values on n, Legendre (1752-1833) [7, 13] is the one in (1809) who introduced the name “Gamma function” and the notation $\Gamma(\alpha)$. However, the relation (1.7), as we will see, does not seem to be recognized by any of them.

In the following theorem we present the following important functional relation that the gamma function has.

Theorem 1.1.3.

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha) \qquad (Re(\alpha) > 0). \tag{1.3}$$

Proof. If we use integration by parts we find that

$$\begin{aligned}\Gamma(\alpha+1) &= \int_0^\infty t^\alpha e^{-t} dt = -e^{-t} t^\alpha \Big|_0^\infty + \int_0^\infty \alpha t^{\alpha-1} e^{-t} dt \\ &= \alpha \int_0^\infty t^{\alpha-1} e^{-t}\, dt = \alpha\,\Gamma(\alpha) \qquad (Re(\alpha) > 0).\end{aligned}$$

□

Lemma 1.1.4.

$$\Gamma(1) = 1. \tag{1.4}$$

Proof. From the definition of the gamma function we have by (1.1) that

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = -e^{-t}\Big|_0^\infty = 1.$$

□

Theorem 1.1.5.

$$\Gamma(n+1) = n!, \qquad (n = 0, 1, 2, \ldots). \tag{1.5}$$

Proof. Setting $\alpha = n$ in (1.3) and iterating the process togethrer with (1.4) we obtain (1.5) as desired. □

Remark: (1.5) provides the relation between the gamma function and the factorial function and also shows how the gamma function is to be looked at as an extension of the factorial function.

Theorem 1.1.6. *The beta function is symmetric on its two variables. That is*

$$\beta(u, v) = \beta(v, u) \tag{1.6}$$

Proof. If we substitute $t = 1 - s$, in (1.2), we get

$$\begin{aligned}\beta(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1}dt &= -\int_1^0 (1-s)^{u-1}s^{v-1}\,ds \\ &= \int_0^1 s^{v-1}(1-s)^{u-1}\,ds \\ &= \beta(v,u).\end{aligned}$$

□

Now, we present the relation between the beta function and the gamma function in the form of the following theorem [2, 11].

Theorem 1.1.7.

$$\beta(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}, \quad (Re(u) > 0,\ Re(v) > 0). \tag{1.7}$$

Proof. First, we make the substitution $t = x^2$ in (1.1) to obtain

$$\Gamma(u) = 2\int_0^\infty e^{-x^2}x^{2u-1}\,dx \quad (Re(u) > 0). \tag{1.8}$$

Multiplying two such integrals together, we find

$$\Gamma(u)\Gamma(v) = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}x^{2u-1}y^{2v-1}\,dx\,dy, \quad (Re(u) > 0,\ Re(v) > 0). \tag{1.9}$$

If we transfer to polar coordinates in the double integral, we find

$$\begin{aligned}\Gamma(u)\Gamma(v) &= 4\int_0^\infty\int_0^{\frac{\pi}{2}} e^{-r^2}r^{2(u+v)-1}(\cos\theta)^{2u-1}(\sin\theta)^{2v-1}\,d\theta\,dr \\ &= 4\left(\int_0^\infty e^{-r^2}r^{2(u+v)-1}\,dr\right)\left(\int_0^{\frac{\pi}{2}}(\cos\theta)^{2u-1}(\sin\theta)^{2v-1}\,d\theta\right).\end{aligned} \tag{1.10}$$

From (1.8) and (1.10), we find

$$\int_0^{\frac{\pi}{2}} (\cos\theta)^{2u-1}(\sin\theta)^{2v-1}\,d\theta = \frac{\Gamma(u)\Gamma(v)}{2\Gamma(u+v)}, \quad (Re(u) > 0,\ Re(v) > 0). \tag{1.11}$$

Substituting $x = \cos^2\theta$ in (1.11), we obtain

$$\int_0^1 x^{u-1}(1-x)^{v-1}\,dx = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}, \quad (Re(u) > 0,\ Re(v) > 0). \tag{1.12}$$

Note that the left hand side is $\beta(u,v)$ as desired. □

The following corollary provides yet another integral representation of the beta function.

Corollary 1.1.8.

$$\beta(u,v) = \int_0^\infty \frac{t^{u-1}}{(1+t)^{u+v}}\,dt, \qquad (Re(u) > 0, Re(v) > 0). \tag{1.13}$$

Proof. If we substitute $t = \frac{x}{1-x}$ in (1.12) we obtain (1.13) as desired. □

1.2 The Extension of the Gamma and Beta Functions

In this section we provide the extension that has been recently made by Chaudhry et.al. on both of the gamma function and the beta function respectively [9, 12].

Definition 1.2.1. *The extended gamma function is defined by*

$$\Gamma_b(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t-b/t}\,dt \qquad (Re(b) > 0; b = 0,\ Re(\alpha) > 0). \tag{1.14}$$

The factor $e^{-\frac{b}{t}}$ in the integral (1.14) plays the role of a regularizer. For $Re(b) > 0, \Gamma_b(\alpha)$ is defined in the complex plane and for $b = 0$, the function $\Gamma_b(\alpha)$ coincides with the classical gamma function.

Definition 1.2.2. *The extended beta function is defined by [9],[11, eq (5.60)]*

$$\beta(u,v;p) = \int_0^1 t^{u-1}(1-t)^{v-1} exp\left[\frac{-p}{t(1-t)}\right] dt \quad (Re(p)>0;\, p=0,\ Re(u)>0, Re(v)>0), \tag{1.15}$$

where the factor $\frac{-p}{t(1-t)}$ again plays the factor of a regularizer of the beta function and at the same time keeps the symmetrical property of the beta function.

Actually, with the choice of the factors of the extension on both functions, most properties of the two functions are preserved where the original properties are deduced as special cases when the parameter p is set to zero. For more details we refer to [9, 11, 12]. For the matter of illustration we present here some results on the gamma function that are available in the literature [11] where the new results we developed in our research work on the beta function would be presented later in a separate section that would follow in the next chapter.

Theorem 1.2.3. *(The Difference Formula)*

$$\Gamma_b(\alpha+1) = \alpha\Gamma_b(\alpha) + b\Gamma_b(\alpha-1). \tag{1.16}$$

Proof. Let M be the Mellin transform operator defined by

$$M\{f(t);\alpha\} := \langle t_+^{\alpha-1}, f(t)\rangle := \int_0^\infty t^{\alpha-1} f(t)\, dt. \tag{1.17}$$

Then, $\Gamma_b(\alpha)$ is simply the Mellin transform of $f(t) = e^{-t-bt^{-1}}$ in α. That is,

$$\Gamma_b(\alpha) := M\left\{e^{-t-bt^{-1}};\alpha\right\}. \tag{1.18}$$

Recalling the relationship,

$$M\{f'(t);\alpha\} = -(\alpha-1)\ M\{f(t);\alpha-1\} \tag{1.19}$$

between the Mellin transform of a function and its derivative, we find

$$-(\alpha-1)\ \Gamma_b(\alpha-1) = M\left\{\left(-1+bt^{-2}\right)e^{-t-bt^{-1}};\alpha\right\}, \tag{1.20}$$

which simplifies to give

$$-(\alpha-1)\Gamma_b(\alpha-1) = -\Gamma_b(\alpha)+b\Gamma_b(\alpha-2). \tag{1.21}$$

Replacing α by $\alpha+1$ in (1.21) we get the proof of (1.16). □

Remark: The functional relation of the classical gamma function presented in (1.3) is obtained as a special case of (1.16) by setting $b=0$.

The following theorem provides another integral representation for the extended beta function in terms of the trigonometric functions.

Theorem 1.2.4.

$$\beta(u,v;p) = 2\int_0^{\frac{\pi}{2}} (\cos\theta)^{2u-1}(\sin\theta)^{2v-1}\, exp\left(-p\csc^2 2\theta\right) d\theta\,. \tag{1.22}$$

Proof. Setting $u=\cos\theta$ and $v=\sin\theta$ in (1.15) gives (1.22) as desired. □

It is time now to provide the connection between the extended gamma function and the extended beta function as in the following theorem.

Theorem 1.2.5. *(Product Formula)*

$$\Gamma_b(u)\Gamma_b(v) = 2\int_0^{\infty} r^{2(u+v)-1}\, exp(-r^2)\beta\left(u,v;\frac{b}{r^2}\right) dr, \tag{1.23}$$

where $\beta(u,v;b)$ is the extended beta function.

Proof. The transformation $t = x^2$ in (1.14) yields

$$\Gamma_b(u) = 2\int_0^\infty x^{2u-1}\, e^{-x^2-bx^{-2}}\, dx \qquad (Re(b) > 0; b = 0, Re(u) > 0). \tag{1.24}$$

Multiplying $\Gamma_b(u)\Gamma_b(v)$ by using (1.24), we find

$$\Gamma_b(u)\Gamma_b(v) = 4\int_0^\infty \int_0^\infty x^{2u-1} y^{2v-1}\, exp\left\{-\left(x^2+y^2\right) - b\left(\frac{x^2+y^2}{x^2y^2}\right)\right\} dx\; dy. \tag{1.25}$$

If we transfer (1.25) to polar coordinates, by setting $x = r\cos\theta$ and $y = r\sin\theta$, we find $\Gamma_b(u)\Gamma_b(v)$

$$= 2\int_0^\infty r^{2(u+v)-1}\, e^{-r^2} \left\{2\int_0^{\frac{\pi}{2}} (\cos\theta)^{2u-1} (\sin\theta)^{2v-1}\, exp\left(\frac{-4b}{r^2}\csc^2 2\theta\right) d\theta\right\} dr. \tag{1.26}$$

By noting that the inner integral in (2.17) is the extended beta function given by (1.22), we obtain (1.23). □

Remark: Although (1.23) generalizes (1.7) where it is obtained by setting $b = 0$ in (1.23), (1.23) does not provide an elegant relation between the extended gamma function and that of the extended beta function as it is in (1.7). This of course would have its effects on the generalized results that we seek to have and on the way that such results are going to be developed as we would see. So, we will try to develop some new results on the extended beta function, which is the basis of our devolvement of the extended Bessel function, that do not depend on the properties of the extended gamma function.

1.3 Some Background on the Bessel Function of the First Kind

Bessel Functions have been known for a long time. In the early eighteenth century, Bessel functions were defined by Daniel Bernoulli (1700-1782) [27] in series form, to discuss the motion of a heavy, oscillating chain. By 1824, Bessel (1784-1846) worked on the equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-\alpha^2)y=0, \tag{1.27}$$

which now bears his name [37]. He derived many properties of its solutions including integral representations. These functions are among the most useful special functions of mathematical physics [34, 36], and are essential in problems possessing cylindrical symmetry [5]. It is to be noted that the Bessel differential equation arises when finding separable solutions to Laplace's equation [4]. Therefore, they are especially important for many problems of wave propagation (telecommunication problems) [22, 24], static potentials and so on [25, 27, 30, 35].

The standard Bessel functions have many forms of representations [1, 17, 38]. Some are more important than others depending on the place where they appear. Bessel functions of the first kind can be defined either from a generating function or from the differential equation that the satisfy [5]. But ultimately both directions will come across each other i.e. (the differential equation would lead to the Bessel function) by means of Frobenius method [8, 27] and (the Bessel function would provides the differential equation) by means of direct differentiation [2]. The Bessel function of the first kind is defined for integral values as

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+n}}{k!\,(n+k)!} \quad (-\infty < x < \infty) \tag{1.28}$$

where the following function

$$w(x,t) = exp\left[\frac{1}{2}x\left(t - \frac{1}{t}\right)\right] \tag{1.29}$$

is its generating function as the following theorem shows.

Theorem 1.3.1.

$$w(x,t) = \sum_{-\infty}^{\infty} J_n(x)t^n. \tag{1.30}$$

Proof.

$$\begin{aligned} w(x,t) &= e^{\frac{xt}{2}} \cdot e^{\frac{-x}{2t}} \\ &= \sum_{j=0}^{\infty} \frac{\left(\frac{xt}{2}\right)^j}{j!} \cdot \sum_{k=0}^{\infty} \frac{\left(\frac{-x}{2t}\right)^k}{k!} \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{j+k} \cdot t^{j-k}}{j!\,k!} \end{aligned} \tag{1.31}$$

Setting $n = j - k$ in (1.31) we have

$$w(x,t) = \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+n}}{k!(n+k)!} t^n. \tag{1.32}$$

So, as $J_n(x)$ is given as in (1.28), we get

$$exp\left(\frac{1}{2}x\left(t - \frac{1}{t}\right)\right) = \sum_{n=-\infty}^{\infty} J_n(x)t^n \quad (t \neq 0) \tag{1.33}$$

as desired. $\square$

It is to be noted here that as

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k-n}}{k!\,(k-n)!}$$
$$= \sum_{k=n}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k-n}}{k!\,(k-n)!} \left(\text{since } \frac{1}{(k-n)!} = 0, k = 0, \ldots, n-1\right), \tag{1.34}$$

then by changing the index by setting $(k = m+n)$ and so $(m = k-n)$ we get

$$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^{m+n} \left(\frac{x}{2}\right)^{2m+n}}{m!\,(m+n)!}$$
$$= (-1)^n \sum_{m=0}^{\infty} \frac{(-1)^{m} \left(\frac{x}{2}\right)^{2m+n}}{m!\,(m+n)!}$$
$$= (-1)^n J_n(x). \tag{1.35}$$

Hence, by (1.35), we see that as n being an integer, J_n and J_{-n} are linearly dependent.

Corollary 1.3.2. *$J_0(0) = 1$ and $J_n(0) = 0$ for $n \neq 0$.*

Proof. As

$$exp\left(\frac{1}{2}x\left(t - \frac{1}{t}\right)\right) = \sum_{n=-\infty}^{\infty} J_n(x)t^n \qquad (t \neq 0),$$

then by setting $x = 0$ we get

$$1 = \sum_{n=-\infty}^{\infty} J_n(0)t^n \qquad (t \neq 0).$$

By equating the coefficients of powers of t we find that $J_0(0) = 1$ and $J_n(0) = 0$ for $n \neq 0$ as desired. □

Indeed, Bessel function could be defined for nonintegral values as the gamma function is the generalization of the factorial function as we have seen in the previous section.

Definition 1.3.3.

$$J_{\alpha}(x)=\sum_{k=0}^{\infty}\frac{(-1)^{k}\left(\frac{x}{2}\right)^{2k+\alpha}}{k!\Gamma(k+\alpha+1)} \tag{1.36}$$

$$J_{-\alpha}(x)=\sum_{k=0}^{\infty}\frac{(-1)^{k}\left(\frac{x}{2}\right)^{2k-\alpha}}{k!\Gamma(k-\alpha+1)} \tag{1.37}$$

Remark: If ($\alpha \neq 0,1,2,\ldots$), then $J_{\alpha}(x)$ and $J_{-\alpha}(x)$ are linearly independent.

The following figure provides the graph of the BF.

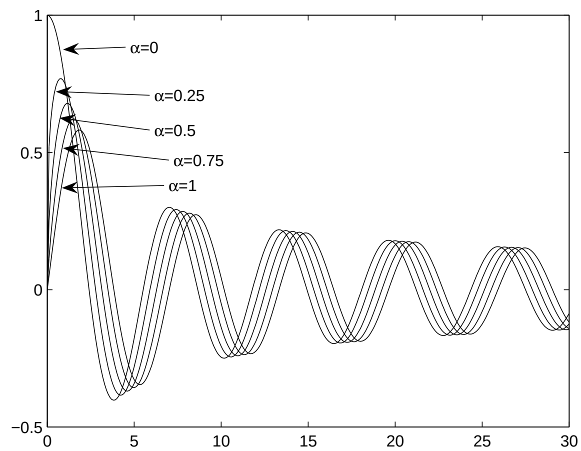

Figure 1.1: Plot of the BF for different values of α.

Theorem 1.3.4.

$$\frac{d}{dx}\left[x^{p}J_{p}(x)\right]=x^{p}J_{p-1}(x). \tag{1.38}$$

Proof. By using (1.36)

$$
\begin{aligned}
\frac{d}{dx}\left[x^p J_p(x)\right] &= \frac{d}{dx}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+2p}}{k!2^{2k+p}\Gamma(k+p+1)} \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k 2(k+p)x^{2k+2p-1}}{k!\,m2^{2kp}\Gamma(k+p+1)} \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+2p-1}}{k!\,2^{2k+p-1}\Gamma(k+p)} \\
&= x^p\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{x}{2}\right)^{2k+p-1}}{k!\,\Gamma(k+p)} \\
&= x^p J_{p-1}(x)
\end{aligned}
$$

as desired. □

Theorem 1.3.5.

$$
\frac{d}{dx}\left[x^{-p}J_p(x)\right] = -x^{-p}J_{p+1}(x). \tag{1.39}
$$

Proof. Again by using (1.36)

$$
\begin{aligned}
\frac{d}{dx}\left[x^{-p}J_p(x)\right] &= \frac{d}{dx}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{k!2^{2k+p}\Gamma(k+p+1)} \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k(2k)x^{2k-1}}{k!2^{2k+p}\Gamma(k+p+1)} \\
&= -x^{-p}\sum_{k=0}^{\infty}\frac{(-1)^{k-1}x^{2k+p-1}}{(k-1)2^{2k+p-1}\Gamma(k+p+1)} \\
&= -x^{-p}\sum_{k=0}^{\infty}\frac{(-1)^{k-}\left(\frac{x}{2}\right)^{2(k-1)+p+1}}{(k-1)!\,\Gamma((k-1)+p+2)} \\
&= -x^{-p}\sum_{m=-1}^{\infty}\frac{(-1)^m\left(\frac{x}{2}\right)^{2m+p+1}}{m!\,\Gamma(m+p+2)} \\
&= -x^{-p}\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{x}{2}\right)^{2m+p+1}}{m!\,\Gamma(m+p+2)} \\
&= -x^{-p}J_{p+1}(x)
\end{aligned}
$$

as desired. $\square$

Corollary 1.3.6.

$$J_p'(x) + \frac{p}{x}J_p(x) = J_{p-1}(x). \tag{1.40}$$

Proof. Differentiating the left side of (1.38) with respect to x gives

$$px^{p-1}J_p(x) + x^p J_p'(x) = x^p J_{p-1}(x).$$

If we divide both sides of the above equation by x^p, (1.40) is obtained. $\square$

Corollary 1.3.7.

$$J_p'(x) - \frac{p}{x}J_p(x) = -J_{p+1}(x). \tag{1.41}$$

Proof. Differentiating the left side of (1.39) gives

$$x^{-p}J_p'(x) + (-p)x^{-p-1}J_p(x) = -x^p J_{p+1}(x).$$

Again, dividing both sides of the above equation by x^p, will give (1.41) as desired. $\square$

Setting $p = 0$ in (1.41) gives the following well known result.

Corollary 1.3.8.

$$J_0'(x) = -J_1(x). \tag{1.42}$$

The next corollary provides three-terms recurrence formula.

Corollary 1.3.9.

$$2J_p'(x) = J_{p-1}(x) - J_{p+1}(x) \tag{1.43}$$

and

$$\frac{2p}{x}J_p(x) = J_{p-1}(x) + J_{p+1}(x) \tag{1.44}$$

Proof. Adding (1.40) with (1.41) gives (1.43) whereas subtracting (1.41) from (1.40) gives (1.44). □

We conclude this section by providing a proof of the Bessel equation given by (1.27) where, in this proof, we used the letter p instead of α appearing in (1.27).

Theorem 1.3.10.

$$x^2 J_p''(x) + xJ_p'(x) + \left(x^2 - p^2\right) J_p(x) = 0\,. \tag{1.45}$$

Proof. We start by rewriting (1.40) as

$$xJ_p'(x) - xJ_{p-1}(x) + pJ_p(x) = 0 \tag{1.46}$$

and differentiate it to get

$$xJ_p''(x) + J_p'(x) = xJ_{p-1}'(x) - J_{p-1}(x) + PJ_p'(x) = 0.$$

Or

$$xJ_p''(x) + (p+1)J_p'(x) - xJ_{p-1}'(x) - J_{p-1}(x) = 0\,. \tag{1.47}$$

Now, by multiplying (1.47) by x and subtracting it from (1.46) we get

$$x^2 J_p''(x) + xJ_p'(x) - p^2 J_p(x) + (p-1)xJ_{p-1}(x) - x^2 J_{p-1}'(x) = 0\,. \tag{1.48}$$

To eliminate $J_{p-1}'(x)$ and $J_{p-1}(x)$ from (1.48), we use (1.41) but after shifting it one step backward to get (1.45) as desired. □

Remark: Above proof goes from the function to the differential equation and shows how the function satisfies the differential equation by direct derivation. However, one could go on the reverse direction by using the method of Frobenius to prove that Bessel function is indeed a solution of (1.27). For the details we refer to [27, 40].

Table 1.1: Some representative values of J_α with different values of α.

x	$\alpha = 0.00$	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 1.00$
0	1	0	0	0	0
0.1	0.9975	0.5207	0.2519	0.1149	0.0499
0.2	0.99	0.6155	0.3545	0.1924	0.0995
0.3	0.9776	0.6743	0.4305	0.2589	0.1483
0.4	0.9604	0.7144	0.4913	0.318	0.196
0.5	0.9385	0.7417	0.541	0.3711	0.2423
0.6	0.912	0.7589	0.5816	0.4187	0.2867
0.7	0.8812	0.7677	0.6144	0.4612	0.329
0.8	0.8463	0.769	0.6399	0.4987	0.3688
0.9	0.8075	0.7637	0.6588	0.5311	0.4059
1	0.7652	0.7522	0.6714	0.5587	0.4401
1.1	0.7196	0.7352	0.678	0.5812	0.4709
1.2	0.6711	0.7129	0.6789	0.5989	0.4983
1.3	0.6201	0.6859	0.6743	0.6116	0.522
1.4	0.5669	0.6545	0.6645	0.6194	0.5419
1.5	0.5118	0.6192	0.6498	0.6225	0.5579
1.6	0.4554	0.5804	0.6305	0.6208	0.5699
1.7	0.398	0.5384	0.6068	0.6145	0.5778
1.8	0.34	0.4937	0.5792	0.6038	0.5815
1.9	0.2818	0.4467	0.5478	0.5889	0.5812
2	0.2239	0.3978	0.513	0.5698	0.5767
2.1	0.1666	0.3475	0.4753	0.5469	0.5683
2.2	0.1104	0.2962	0.4349	0.5204	0.556
2.3	0.0555	0.2444	0.3923	0.4906	0.5399
2.4	0.0025	0.1923	0.3479	0.4578	0.5202
2.5	-0.0484	0.1406	0.302	0.4223	0.4971
2.6	-0.0968	0.0895	0.2551	0.3844	0.4708
2.7	-0.1424	0.0394	0.2075	0.3445	0.4416
2.8	-0.185	-0.0092	0.1597	0.3029	0.4097
2.9	-0.2243	-0.056	0.1121	0.26	0.3754
3	-0.2601	-0.1006	0.065	0.2162	0.3391

Chapter 2

The Extended Complete and Incomplete Beta Functions

As stated in the introductory chapter that the beta function which is also called Euler integral of the first kind and was introduced by Euler plays an important role in the study of some important special functions [4, 16, 27, 34, 36]. It can be written in terms of the gamma function as by (1.3)

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

and has the integral representation (1.2)

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt \quad (Re(x) > 0, Re(y) > 0).$$

The gamma functions, themselves can be decomposed into the two incomplete gamma functions [12]:

$$\gamma(\alpha,x) = \int_0^x t^{\alpha-1}\, exp\,(-t)dt \quad (Re(\alpha) > 0, |arg(\alpha) < \pi|), \tag{2.1}$$

$$\Gamma(\alpha,x) = \int_x^\infty t^{\alpha-1}\, exp\,(-t)dt \quad (|arg(\alpha) < \pi|), \tag{2.2}$$

which led to the definition of the incomplete beta function

$$\beta_t(x,y) = \int_0^t h^{x-1}(1-h)^{y-1}dh \quad (Re(x)>0, Re(y)>0 \, and \, 0 \leq t \leq 1). \tag{2.3}$$

This function was extended by introducing a new parameter in the integral representation given by (1.15)

$$\beta(x,y;p) = \int_0^1 t^{x-1}(1-t)^{y-1} exp\left[\frac{-p}{t(1-t)}\right] dt \quad (Re(p)>0).$$

The extension cannot be simply expressed in terms of the generalized gamma function as in (1.3). For $p=0$, this extension reduces to the standard beta function. The extension caries through trivially to the incomplete beta function as

$$\beta_t(x,y;p) = \int_0^t h^{x-1}(1-h)^{y-1} exp\left[\frac{-p}{h(1-h)}\right] dh \quad (Re(p)>0). \tag{2.4}$$

For more details we refer to [9, 11]. As an example where such an extension was applied, is the extension of the Gauss hypergeometric function [10] which is defined as

$$F_p(a,b,c,z) := \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{n=0}^{\infty} \frac{(a)_n \beta(b+n, c-b; p) z^n}{n!}$$

$$(p \geq 0; |z|<1; Re(c) > Re(b) > 0). \tag{2.5}$$

As such, it should be useful to investigate other properties of the extended beta function.

In this chapter, we derive further properties relating these extended beta function to the integral of its incomplete version. These relations lead to relations satisfied by the extended beta function, including the relationship that the difference between $\beta(x+n,y;p)$ and $\beta(x+1,y;p)$ is the same as the corresponding difference when the shift is applied in the second variable. This result holds even for the standard beta function.

In section two of this chapter, we give some general formulae that involve the incomplete extended beta function. Then, by taking special cases of the general forms, some important and new relations are given. Also, a generalization of the well known functional property of the standard and extended beta function is given. In section three, we apply results of section two where new relations are found. One relation is between the complete version of the extended beta function. The other is an elegant relationship that justifies the difference between two extended beta functions with any integral shift in one of the variables as just described in the above paragraph. Finally, we finish the chapter with a concluding section.

2.1 Integrals Involving the Incomplete Extended Beta Function

As the independent variable of the incomplete extended beta function $t \in [0,1]$, the integrals of these functions are found over the unit interval. The following theorem shows that they can be expressed in closed forms.

Theorem 2.1.1.

$$\int_0^1 t^{s-1}\beta_t(x,y;p)dt = \frac{1}{s}\left[\beta(x,y;p) - \beta(x+s,y;p)\right] \quad (s \neq 0). \tag{2.6}$$

Proof. We know that

$$\beta(x+s,y;p) = \int_0^1 t^{x+s-1}(1-t)^{y-1} exp\left[\frac{-p}{t(1-t)}\right] dt.$$

Taking t^s as u and the remaining part as dv, we get

$$v = \int_0^t h^{x-1}(1-h)^{y-1} exp\left[\frac{-p}{h(1-h)}\right] dh$$

which is clearly $\beta_t(x,y;p)$. Hence we have

$$\beta(x+s,y;p) = \beta(x,y;p) - s\int_0^1 t^{s-1}\beta_t(x,y;p)dt. \tag{2.7}$$

By rearranging the terms we get the result. □

Taking $s = 1$ in (2.6) we get the following corollary:

Corollary 2.1.2.

$$\int_0^1 \beta_t(x,y;p)dt = \beta(x,y;p) - \beta(x+1,y;p). \tag{2.8}$$

Note that the right side of (2.8) appears as a closed form recurrence relation. From this one can see that a shift of the first variable by one is actually as if we take a quantity from the original extended beta that equals to the integral of incomplete extended beta over the unit interval. Of course Theorem 2.1.1 could be rewritten so as to show the functional relation of the extended beta in its first variable if s is taken to be as an integer k to give

$$\beta(x,y;p) - \beta(x+k,y;p) = k\int_0^1 t^{k-1}\beta_t(x,y;p)dt. \tag{2.9}$$

In contrast with Theorem 2.1.1 where the weighted function is t^{s-1}, the weighted function in the following theorem is $(1-t)^{s-1}$.

Theorem 2.1.3.

$$\int_0^1 (1-t)^{s-1}\beta_t(x,y;p)dt = \frac{1}{s}\beta(x,y+s;p) \quad (s \neq 0). \tag{2.10}$$

Proof. Again, we start with the integral form of $\beta(x, y+s; p)$ and integrate it by parts by setting $u = (1-t)^s$ and the rest as dv. □

Taking $s = 1$ in Theorem 2.1.3 we get:

Corollary 2.1.4.

$$\int_0^1 \beta_t(x, y; p)dt = \beta(x, y+1; p). \tag{2.11}$$

Note that the right side again appears in closed form in a simple and elegant result. In comparison to (2.8), the shift in the second variable by one is just the integral of incomplete extended beta over the unit interval. Indeed one can use (2.7) and (2.10) to get a more general form over the first and second variables of the extended beta.

Theorem 2.1.5.

$$\beta(x+u, y+v; p) = \beta(x, y+v; p) - u\int_0^1 t^{u-1}\beta_t(x, y+v; p)dt \quad (u \neq 0). \tag{2.12}$$

$$\beta(x+u, y+v; p) = v\int_0^1 (1-t)^{v-1}\beta_t(x+u, y; p)dt \quad (v \neq 0). \tag{2.13}$$

Proof. Replace y by $y+v$ in (2.7) to give (2.12) and x by $x+u$ in (2.10) to give (2.13). □

In the following corollary, the extended beta function is written as the integral of an infinite sum of incomplete beta function.

Corollary 2.1.6.

$$\beta(x, y; p) = \int_0^1 \sum_{n=0}^{\infty} \beta_t(x+n, y; p)dt. \tag{2.14}$$

Proof. In (2.13), put $u = n$ and $v = 1$ to get

$$\beta(x+n, y+1; p) = \int_0^1 \beta_t(x+n, y; p)dt.$$

Summing both sides and noting that the summation of the left side is simply the extended beta function [11, p.223 (5.72)], we get (2.14), as the order of summation and integration can be reversed as the integral is uniformly convergent. □

Corollary 2.1.7. (Connection with the Gauss hypergeometric function)

$$2^{-2x}\int_0^1 e^{-4pt}\,{}_2F_1\left(\frac{-m}{2},\frac{1-m}{2},1-m;t\right)t^{x-1}(1-t)^{-1/2}dt$$

$$= m\int_0^1 (1-t)^{m-1}\beta_t(x,x;p)dt. \tag{2.15}$$

Proof. Put $v=n$, $u=0$ and $y=x$ in (2.13) to get

$$\beta(x,x+m;p) = m\int_0^1 (1-t)^{m-1}\beta_t(x,x;p)dt \quad (m\neq 0). \tag{2.16}$$

Also, by [11, p.237 (5.146)] we have:

$$\beta(x,x+m;p) = 2^{-2x}\int_0^1 e^{-4pt}\,{}_2F_1\left(\frac{-m}{2},\frac{1-m}{2},1-m;t\right)$$

$$\times t^{x-1}(1-t)^{-1/2}dt. \tag{2.17}$$

Now, by (2.16) and (2.17), we have (2.15). □

Theorem 2.1.8.

$$\int_0^1 \beta_t(x,y;p)[v(1-t)^{v-1}-ut^{u-1}]dt$$

$$=\beta(x+u,y;p)+\beta(x,y+v;p)-\beta(x,y;p) \quad (u,v\neq 0). \tag{2.18}$$

Proof. Set $s=u$ in (2.7), $s=v$ in (2.10) and add the results together. Then, rearrange the common terms to get (2.18). □

2.2 Applications to the Extended Beta Function

In this section we apply some results derived in the previous section. In particular, we apply Theorem 2.1.8 to obtain a relation between the complete version of the extended beta function as the following theorem says.

Theorem 2.2.1.

$$\beta(x,y;p) = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \beta(x+n-k, y+k; p). \tag{2.19}$$

Proof. In (2.18) put $u = v = 1$ to obtain the well known result

$$\beta(x+1,y;p) + \beta(x,y+1;p) = \beta(x,y;p). \tag{2.20}$$

Now, use the shift iteratively to obtain (2.19). □

Note that the result (2.19) applies also for $p = 0$ and hence it holds for the standard beta function and does not seem to appear in the literature. In the following theorem we obtain another remarkably elegant functional relation for the extend beta function.

Lemma 2.2.2.

$$\beta(x+2,y;p) - \beta(x+1,y;p) = \beta(x,y+2;p) - \beta(x,y+1;p). \tag{2.21}$$

Proof. Take $u = v = 2$ in (2.18) to get

$$\beta(x+2,y;p) + \beta(x,y+2;p) = \beta(x,y;p) + 2\int_0^1 \beta_t(x,y;p)(1-2t)dt. \tag{2.22}$$

Now, by using (2.7) with $s = 2$, (2.22) can be written as

$$\beta(x,y+2;p) - \beta(x+2,y;p) = \beta(x,y;p) - 2\beta(x+1,y;p). \tag{2.23}$$

By applying (2.20) to $\beta(x,y;p)$ appearing on the right side of (2.23) and rearranging the terms, we get (2.21) as desired. □

Theorem 2.2.3.

$$\beta(x+n,y;p) - \beta(x+1,y;p) = \beta(x,y+n;p) - \beta(x,y+1;p) \quad (n \geq 1). \tag{2.24}$$

Proof. We prove the result by induction. First, for $n = 1$, the result holds trivially. For $n = k$, it gives the following true by assumption. That is

$$\beta(x+k,y;p) - \beta(x+1,y;p) = \beta(x,y+k;p) - \beta(x,y+1;p) \quad (n \geq 1). \tag{2.25}$$

For $n = k+1$, we have

$$\beta(x+k+1,y;p) - \beta(x+1,y;p) = \beta((x+1)+k,y;p) - \beta(x+1,y;p). \tag{2.26}$$

We add and subtract $\beta(x+2,y;p)$ to the right side of (2.26) to get

$$\beta(x+k+1,y;p) - \beta(x+1,y;p)$$

$$= [\beta((x+1)+k,y;p) - \beta(x+2,y;p)] + [\beta(x+2,y;p) - \beta(x+1,y;p)]. \tag{2.27}$$

The first bracket in (2.27) gives $\beta(x,y+k+1;p)-\beta(x,y+2;p)$ by (2.25), the induction step and the second bracket gives $\beta(x,y+2;p)-\beta(x,y+1;p)$ by Lemma 2.2.2. So,

$$\beta(x+k+1,y;p)-\beta(x+1,y;p)=\beta(x,y+k+1;p)-\beta(x,y+1;p)$$

as desired and hence the induction proof is complete. $\square$

Again note that this result holds for the standard beta function and does not seem to appear in the literature.

2.3 Concluding Remarks and Observations

We have provided closed forms of some integrals involving the incomplete extended beta function. Moreover, a generalization of an earlier recurrence relation (2.20) appearing in the literature has also been given. It is to be remarked here that in general the generalizations provided do not require the shifts to be integers. It is to be noted also that although we were not able to express $\beta_t(x,y;p)$ in terms of other special functions, we were able to connect it with the Gauss hypergeometric function.

Moreover, we were able to apply our results in finding two new relations for the extended beta function. One relation is between the complete version of the extended beta while the other shows that the difference between the function with the first variable shifted by any integer $n \geq 1$ and that of the function with the first variable shifted by one is the same as the corresponding difference for the second variable. These two results hold fully for the standard beta function and do not seem to be available in the literature.

Chapter 3

Extension of Bessel and Modified Bessel Functions

As we stated in the introduction that the Bessel function has been known since the eighteenth century and it has different representations especially in the integral forms. Examples include:

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\Phi - x\sin\Phi)\,d\Phi\,, \tag{3.1}$$

and when $n = 0$ we have

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\phi)\,d\phi\,. \tag{3.2}$$

For a more general form where p need not to be an integer, we have

$$J_p(x) = \frac{(x/2))^p}{\sqrt{\pi}\,\Gamma\left(p+\frac{1}{2}\right)}\int_{-1}^{1}(1-t^2)^{p-\frac{1}{2}}e^{ixt}\,dt \qquad \left(p > -\frac{1}{2},\, x > 0\right). \tag{3.3}$$

In this chapter we provide a new integral representation that seems not to be recognized in the literature and then develop some new results on such a function. However, our study will go through that of the confluent hypergeometric function from which the Bessel function is found.

The Gauss hypergeometric function (GHF)

$$ {}_2F_1(a,b;c;z) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} \quad (c \neq 0,-1,-2,...,) \tag{3.4} $$

which could be rewritten in terms of the beta function as

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{n=0}^{\infty} (a)_n \beta(b+n,c-b)\frac{z^n}{n!} $$

$$ (Re(c) > Re(b) > 0) \tag{3.5} $$

(see proof of Theorem (3.1.1) for details) has been extended by Chaudhry et. al. [9, 10, 11] by using their extension to the beta function defined before in (1.15). This extension of the GHF is given in (2.5) as

$$ F_p(a,b;c;z) := \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{n=0}^{\infty} \frac{(a)_n \beta(b+n,c-b;p) z^n}{n!} $$

$$ (p \geq 0; |z| < 1; Re(c) > Re(b) > 0). $$

Similarly, the confluent hypergeometric function (CHF)

$$ {}_1F_1(b;c;z) := \sum_{n=0}^{\infty} \frac{(b)_n}{(c)_n}\frac{z^n}{n!} \quad (c \neq 0,-1,-2,...,), \tag{3.6} $$

which could be rewritten as

$$ {}_1F_1(b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{n=0}^{\infty} \beta(b+n,c-b)\frac{z^n}{n!} $$

$$ (Re(c) > Re(b) > 0), \tag{3.7} $$

is extended as

$$\Phi_p(b;c;z) := \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{n=0}^{\infty} \frac{\beta(b+n,c-b;p)z^n}{n!}$$

$$(p \geq 0; Re(c) > Re(b) > 0). \tag{3.8}$$

The functions $F_p(a,b;c;z)$ which converges for $|z| < 1$ and $\Phi_p(b;c;z)$ which converges for all z will be denoted by EGHF and ECHF respectively where the letter "E" stands for the word "extended". By setting $p = 0$, these two functions reduce to the standard functions. These functions were expected to prove useful because of the utility of the original functions. In particular, the CHF contains the special case of ${}_0F_1(-;c;z)$, which is related to the Bessel function of the first kind (BF) [2, 16] by

$$J_\alpha(z) = \frac{(z/2)^\alpha}{\Gamma(\alpha+1)} {}_0F_1(-;\alpha+1;-\frac{z^2}{4}) \qquad (Re(\alpha) > -1). \tag{3.9}$$

and to the modified Bessel function (MBF) by

$$I_\alpha(z) = \frac{(z/2)^\alpha}{\Gamma(\alpha+1)} {}_0F_1\left(-;\alpha+1;\frac{z^2}{4}\right) \quad (Re(\alpha) > -1). \tag{3.10}$$

This special case has the series representation [37]:

$${}_0F_1(-;c;z) = \sum_{n=0}^{\infty} \frac{1}{(c)_n} \frac{z^n}{n!} \quad (c \neq 0,-1,-2,...,) \tag{3.11}$$

where $(c)_n$ in known as the Pochhammer symbol which is defined as

$$(c)_0 = 1, \ (c)_n = c(c+1)\cdots(c+n-1) \quad (n = 1,2,3,\cdots). \tag{3.12}$$

The function ${}_0F_1(-;c;z)$ is easily seen to converge for all z [32]. The BF which is related to ${}_0F_1(-;c;z)$ as in (3.9) has the following series representation [37]:

$$J_\alpha(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n+\alpha}}{n!\Gamma(\alpha+n+1)} \quad (Re(\alpha) > -1). \tag{3.13}$$

This function has been generalized in various ways and the properties of these generalized functions have been studied [14, 15]. In this work, however, we present a different generalization along the lines of the extension of the CHF. Although our generalization does not enlarge the domain of the newly generalized function (as we will see), we call it an extended one because it is defined in terms of the extended beta function and to avoid over-utilizing the term “generalized” [14, 15].

In this work we have extended the confluent (incomplete) hypergeometric function and hence obtained results for the modified Bessel function. *One of these results give a formula for the standard Bessel function relating the Bessel of any order to the integral of the Bessel of first order.*

This chapter is organized as follows. In section two, a new integral representation for the CHF is developed. This leads to a new integral representation of the BF and the MBF. In this new representation, *all the BFs and the MBFs are found from the BF and the MBF of order one respectively.* In section three, we define the extended confluent hypergeometric function (ECHF) and give some of its important properties. The subsequent section will define the extended BF (EBF) and the extended MBF (EMBF) and provides some of their properties. In this section, an integral representation is developed for the EBF as well as for the EMBF. In section 5, we define the incomplete CHF and the incomplete BF and their extensions. We also give some results on that issue by relating the incomplete version with the complete version through finding the integral of the incomplete version over the unit interval. In section 6, we give graphs of the EBF and that of the EMBF. We finish this work by concluding remarks.

It is to be remarked that whenever we use the symbols (BF) and (EBF) in this chapter and hereafter, we mean by that the *standard and the extended Bessel functions of the first kind* respectively.

3.1 An Integral Representation for the CHF and the BF

There are different integral representations for ${}_0F_1(-;c;z)$ in the literature [19, 20, 21]. However, we present, in the following theorem, another integral representation which is needed for our generalization of the BF.

Theorem 3.1.1.

$$ {}_0F_1(-;c;z) = 1 + z\int_0^1 (1-t)^{c-1}\,{}_0F_1(-;2;tz)dt. \tag{3.14} $$

Proof. Using

$$ (c)_n = \frac{\Gamma(n+c)}{\Gamma(c)} $$

and so

$$ \beta(n,c) = \frac{\Gamma(n)\Gamma(c)}{\Gamma(n+c)} = \frac{\Gamma(n)}{\Gamma(n+c)/\Gamma(c)} = \frac{\Gamma(n)}{(c)_n} $$

which gives

$$ \frac{1}{(c)_n} = \frac{\beta(n,c)}{\Gamma(n)}, $$

(3.11) can then be rewritten as

$$ {}_0F_1(-;c;z) = \sum_{n=0}^{\infty} \frac{\beta(n,c)z^n}{\Gamma(n)\,n!}. \tag{3.15} $$

The first term in (3.15) is one. Putting $n \to n+1$, it can be written as

$$ {}_0F_1(-;c;z) = 1 + z\sum_{n=0}^{\infty} \frac{\beta(n+1,c)}{(2)_n}\ \frac{z^n}{n!} \tag{3.16} $$

where

$$ \Gamma(n+2) = (n+1)! = (1)_{n+1} = (2)_n\,. $$

We now use the integral representation of the beta function given by (1.2) to get

$$ {}_0F_1(-;c;z) = 1 + z\int_0^1 (1-t)^{c-1}\left(\sum_{n=1}^{\infty} \frac{t^n z^n}{(2)_n n!}\right) dt \quad (c>0). \tag{3.17} $$

The series in (3.17) is nothing but ${}_0F_1(-;2;tz)$. Hence, we have (3.14). □

Corollary 3.1.2.

$$ {}_0F_1(-;c;z) = 1 + z\int_0^{\infty} \frac{1}{(1+u)^{c+1}}\,{}_0F_1\left(-;2;\frac{uz}{1+u}\right) du. \tag{3.18} $$

Proof. In (3.14) use the transformation $t = \frac{u}{1+u}$. Then, (3.18) is obtained by simplifying the resultant integral. □

The following theorem gives a new integral representation for the standard BF.

Theorem 3.1.3.

$$ J_\alpha(z) = \frac{(z/2)^\alpha}{\Gamma(\alpha+1)}\left[1 - \frac{z}{2}\int_0^1 \frac{(1-t)^\alpha}{\sqrt{t}} J_1(z\sqrt{t})dt\right] \quad (Re(\alpha) > -1). \tag{3.19} $$

Proof. If we replace $c \to \alpha+1$ and $z \to -\frac{z^2}{4}$ in (3.14) we get

$$ {}_0F_1\left(-;\alpha+1;-\frac{z^2}{4}\right) = 1 - \frac{z^2}{4}\int_0^1 (1-t)^\alpha\,{}_0F_1\left(-;2;-\frac{1}{4}(z\sqrt{t})^2\right) dt. \tag{3.20} $$

But since by (3.9)

$$J_\alpha(z) = \frac{(z/2)^\alpha}{\Gamma(\alpha+1)}\, {}_0F_1\left(-;\alpha+1;-\frac{z^2}{4}\right),$$

we have

$${}_0F_1\left(-;2;-\frac{1}{4}(z\sqrt{t})^2\right) = \frac{2}{z\sqrt{t}} J_1(z\sqrt{t}). \tag{3.21}$$

By substituting (3.21) in (3.20) we get

$${}_0F_1\left(-;\alpha+1;-\frac{z^2}{4}\right) = 1 - \frac{z}{2}\int_0^1 \frac{(1-t)^\alpha}{\sqrt{t}} J_1(z\sqrt{t})dt. \tag{3.22}$$

Putting (3.22) in (3.9) again, we get (3.19) as desired. □

It is worth mentioning here that by using this representation, $J_\alpha(.)$ *can simply be obtained by a transformation of* $J_1(.)$. Although the BFs have been studied extensively, this representation does not seem to have appeared in the literature in the present form.

Corollary 3.1.4.

$$J_\alpha(z) = \frac{(z/2)^\alpha}{\Gamma(\alpha+1)}\left[1 - \int_0^z \left(1-\frac{u^2}{z^2}\right)^\alpha J_1(u)du\right] \quad (Re(\alpha) > -1). \tag{3.23}$$

Proof. By using the transformation $u = z\sqrt{t}$ in (3.18), we get (3.23). □

Corollary 3.1.5.

$$J_0(z) = 1 - \int_0^z J_1(u)du. \tag{3.24}$$

Proof. Put $\alpha = 0$ as a special case of (3.23) to get (3.24). □

Remark: Note that the well known relation in the literature [2, 5, 16, 37]

$$J_0'(z) = -J_1(z), \tag{3.25}$$

could be integrated to give (3.24) under the condition that $J_0(0)=1$. In this sense, Corollary (3.1.5) generalizes this relation (3.25).

Corollary 3.1.6.

$$J_\alpha(z) = 1 - z\int_0^{\frac{\pi}{2}} (\cos\theta)^{2\alpha+1} J_1(z\sin\theta)\,d\theta. \tag{3.26}$$

Proof. If we use the transformation $u = z\sin\theta$ in (3.23), we get (3.26). □

We can compare this result (3.26) with the following result appearing in the literature [37, 12.12]

$$\int_0^{\frac{\pi}{2}} J_\mu(z\sin\theta)(\sin\theta)^{\mu+1}(\cos\theta)^{2\nu+1}d\theta = \frac{1}{2}\Gamma(\nu+1)\left(\frac{2}{z}\right)^{\nu+1} J_{\mu+\nu+1}(z). \tag{3.27}$$

We see that restriction of parameters in (3.27) will not imply (3.26). However, in (3.26) we found another general formula in a simple form that involves the BF and trigonometric functions.

Now, we provide the following results on the MBF. We skip the proofs because of their similarity to that provided for the BF.

Theorem 3.1.7.

$$I_\alpha(z) = \frac{(z/2)^\alpha}{\Gamma(\alpha+1)}\left[1 + \frac{z}{2}\int_0^1 \frac{(1-t)^\alpha}{\sqrt{t}} I_1(z\sqrt{t})dt\right] \quad (Re(\alpha) > -1). \tag{3.28}$$

It is worth mentioning here that by using this representation, $I_\alpha(.)$ *can simply be obtained by transforming* $I_1(.)$. Again, this representation does not seem to have appeared in the literature as in the present from.

Corollary 3.1.8.

$$I_\alpha(z) = \frac{(z/2)^\alpha}{\Gamma(\alpha+1)}\left[1+\int_0^z \left(1-\frac{u^2}{z^2}\right)^\alpha I_1(u)du\right] \quad (Re(\alpha)>-1). \tag{3.29}$$

Corollary 3.1.9.

$$I_0(z) = 1+\int_0^z I_1(u)du. \tag{3.30}$$

Remark: Note that the well known relation in the literature [2, 5, 16, 37]

$$I_0^{'}(z) = I_1(z), \tag{3.31}$$

could be obtained by direct differentiation of (3.30).

Corollary 3.1.10.

$$I_\alpha(z) = 1+z\int_0^{\frac{\pi}{2}} (\cos\theta)^{2\alpha+1} I_1(z\sin\theta)d\theta. \tag{3.32}$$

3.2 Definition and Properties of Extended ${}_0F_1(-;c;z;p)$

By following same procedure as for defining the EGHF, we define extended ${}_0F_1$ as follows:

$${}_0F_1(-;c;z;p) = 1+z\sum_{n=0}^{\infty}\frac{\beta(n+1,c;p)}{(2)_n}\frac{z^n}{n!}. \tag{3.33}$$

It is to be noted here and hereafter that $(p \geq 0)$. Of course for $p=0$, we are back to standard cases and the standard conditions follow. Actually, as in the standard ${}_0F_1$, the series in (3.33) converges for all z. This is proved in Theorem 3.2.2. For that purpose we prove the following lemma.

Lemma 3.2.1.

$$\lim_{n\to\infty}\left|\frac{\beta(n+2,c;p)}{\beta(n+1,c;p)}\right| = 1. \tag{3.34}$$

Proof. For fixed p and y, the extended beta function has the following asymptotic behavior for large value of x[10]:

$$\beta(x,y;p) \sim \frac{c}{e^{2\sqrt{p}\sqrt{x}}x^{3/4}\sqrt{x^{(y-1)}}}$$

where c is a constant. Hence, by (A.1) we have

$$\lim_{n\to\infty}\left|\frac{\beta(n+2,c;p)}{\beta(n+1,c;p)}\right| = \lim_{n\to\infty}\left|\frac{e^{2\sqrt{p}\sqrt{n}}n^{3/4}\sqrt{n^{(y-1)}}}{e^{2\sqrt{p}\sqrt{n+1}}\,(n+1)^{3/4}\sqrt{(n+1)^{(y-1)}}}\right| = 1$$

as desired. $\square$

Theorem 3.2.2.

$${}_0F_1(-;c;z;p) = 1 + z\sum_{n=0}^{\infty}\frac{\beta(n+1,c;p)}{(2)_n}\frac{z^n}{n!}.$$

given by (3.33) converges for all z.

Proof. By using the ratio test, we have

$$\lim_{n\to\infty}\left|\frac{\beta(n+2,c;p)z^{n+1}}{(2)_{n+1}(n+1)!}\cdot\frac{n!(2)_n}{\beta(n+1,c;p)z^n}\right|$$

$$= |z|\lim_{n\to\infty}\left|\frac{\beta(n+2,c;p)n!(2)_n}{\beta(n+1,c;p)(n+2)(2)_n(n+1)n!}\right|. \tag{3.35}$$

By (3.34), (3.35) equals zero and so we have convergence for all z. $\square$

Theorem 3.2.3. *(Uniform convergence of the extended ${}_0F_1$)*

The extended ${}_0F_1$ is uniformly convergent for $|z| < 1$.

Proof. From (3.33) we see that it is enough to consider the uniform convergence of this series

$$\sum_{n=0}^{\infty} \frac{\beta(n+1,c;p)}{(2)_n} \frac{z^n}{n!}. \tag{3.36}$$

Since

$$\begin{aligned} |\beta(n+1,c)| &= \left| \frac{\Gamma(n+1)\Gamma(c)}{\Gamma(n+1+c)} \right| = \left| \frac{n!\Gamma(c)}{(n+c)\Gamma(n+c)} \right| \\ &= \cdots = \left| \frac{n!\Gamma(c)}{(n+c)(n+c-1)\cdots(c)\Gamma(c)} \right| \leq \left| \frac{1}{c} \right| \end{aligned} \tag{3.37}$$

and also by using (A.3) we have

$$|\beta(n+1,c;p)| \leq e^{-4p} |\beta(n+1,c)|,$$

then for $|z| < 1$ we get

$$\begin{aligned} \left| \sum_{n=0}^{\infty} \frac{\beta(n+1,c;p)}{(2)_n} \frac{z^n}{n!} \right| &\leq \sum_{n=0}^{\infty} \left| \frac{\beta(n+1,c;p)}{(2)_n} \frac{z^n}{n!} \right| < \sum_{n=0}^{\infty} \left| \frac{\beta(n+1;c;p)}{(2)_n n!} \right| \\ &\leq e^{-4p} \sum_{n=0}^{\infty} \left| \frac{\beta(n+1,c)}{(2)_n n!} \right| \leq \frac{e^{-4p}}{|c|} \sum_{n=0}^{\infty} \left| \frac{1}{(2)_n n!} \right| \\ &= \frac{e^{-4p}}{|c|} \sum_{n=0}^{\infty} \frac{1}{(2)_n n!} = \frac{e^{-4p}}{|c|} \, {}_0F_1(-;2;1) \\ &= M(constant). \end{aligned}$$

Hence our result is proved by the Weierstrass M-test (see Theorem A.1 in the appendix). □

The following theorem gives an integral representation for the extended ${}_0F_1$.

Theorem 3.2.4.

$$ {}_0F_1(-;c;z;p) = 1 + z\int_0^1 (1-t)^{c-1} exp\left[\frac{-p}{t(1-t)}\right] {}_0F_1(-;2;tz)dt. \tag{3.38}$$

Proof. Starting with (3.33) and representing the extended beta by its integral representation, we get

$$ {}_0F_1(-;c;z;p) = 1 + z\sum_{n=1}^{\infty}\int_0^1 t^n(1-t)^{c-1}\frac{z^n}{(2)_n n!} exp\left[\frac{-p}{t(1-t)}\right] dt$$

$$ = 1 + z\int_0^1 (1-t)^{c-1} exp\left[\frac{-p}{t(1-t)}\right]\sum_{n=1}^{\infty}\frac{t^n z^n}{(2)_n n!}dt. \tag{3.39}$$

The series in (3.39) is nothing but ${}_0F_1(-;2;tz)$. Hence, (3.38) is obtained. □

Note that again the ECHF is written in terms of the standard CHF.

Corollary 3.2.5.

$$ {}_0F_1(-;c;z;p)$$

$$ = 1 + ze^{-2p}\int_0^{\infty}\frac{1}{(1+u)^{c+1}} exp\left[-p\left(u+\frac{1}{u}\right)\right] {}_0F_1\left(-;2;\frac{uz}{1+u}\right) du$$

$$ (p > 0; p = 0, Re(c) > 0). \tag{3.40}$$

Proof. In (3.38) use the transformation $t = \frac{u}{1+u}$ and simplify the resultant integral to obtain (3.40). □

The following theorem shows that ${}_0F_1(-;c;z;p)$ is exponentially damped as a function of p.

Theorem 3.2.6.

$$a)\, {}_0F_1(-;c;z;p) \leq 1 + e^{-4p}({}_0F_1(-;c;z) - 1)\ \ (z, c \in \mathscr{R}\, and\, p \in \mathscr{R}^{+}). \tag{3.41}$$

$$b)\, |{}_0F_1(-;c;z;p)| \leq \left|1 + e^{-4p}({}_0F_1(-;c;z) - 1)\right|\ (c, z \in \mathbb{C}\, and\, p \in \mathscr{R}^{+}). \tag{3.42}$$

Proof. Consider $f(u) = exp\left[-p\left(1+\frac{1}{u}\right)\right]$ appearing in (3.40). By (A.5), this function attains its maximum at $u = 1$. Looking back to (3.40), we see that for real values of z and c with $p > 0$, the following inequality

$$ {}_0F_1(-;c;z;p) \leq 1 + ze^{-4p} \int_0^\infty \frac{1}{(1+u)^{c+1}}\, {}_0F_1\left(-;2;\frac{uz}{1+u}\right) du \tag{3.43}$$

holds. Now, by (3.18) we have

$$\int_0^\infty \frac{1}{(1+u)^{c+1}}\, {}_0F_1\left(-;2;\frac{uz}{1+u}\right) du = {}_0F_1(-;c;z) - 1. \tag{3.44}$$

So, (3.41) is obtained by substituting (3.44) in (3.43).

For complex values of z, however, we proceed as follows. Take the absolute value of both sides of (3.40). Then, we have

$$|{}_0F_1(-;c;z;p)| \leq \left|1 + ze^{-4p} \int_0^\infty \frac{1}{(1+u)^{c+1}}\, {}_0F_1\left(-;2;\frac{uz}{1+u}\right) du\right|$$

since $exp\left[-p\left(1+\frac{1}{u}\right)\right]$ attains its maximum at $p = 1$. But by (3.44) again, (3.42) is obtained.

$\square$

3.3 Definition and some Properties of the EBF

As mentioned before, the extension of ${}_0F_1$ will lead to the extension of J_α. So, as by (3.9):

$$J_\alpha(z) = \frac{(z/2)^\alpha}{\Gamma(\alpha+1)}\, {}_0F_1\left(-;\alpha+1;-\frac{z^2}{4}\right)\ (Re(\alpha) > -1),$$

we naturally define the EBF $J_{\alpha,p}(z)$ as:

$$J_{\alpha,p}(z) = \frac{(z/2)^\alpha}{\Gamma(\alpha+1)}\, {}_0F_1\left(-;\alpha+1;-\frac{z^2}{4};p\right)\ (Re(\alpha) > -1), \tag{3.45}$$

then, give the series representation of this EBF. From that we extend some results appearing in the literature for the BF [2, 5, 16, 37] to the EBF.

Theorem 3.3.1. *(Series representation)*

$$J_{\alpha,p}(z) = \frac{(z/2)^\alpha}{\Gamma(\alpha+1)} - \sum_{n=0}^{\infty} \frac{\beta(n+1;\alpha+1;p)(-1)^n z^{2n+\alpha+2}}{\Gamma(\alpha+1)2^{2n+\alpha+2}(2)_n n!}$$

$$(Re(\alpha) > -1). \tag{3.46}$$

Proof. As, by (3.33), the series representation of ${}_0F_1(-;c;z;p)$ is given as

$${}_0F_1(-;c;z;p) = 1 + z\sum_{n=0}^{\infty} \frac{\beta(n+1,c;p)}{(2)_n}\frac{z^n}{n!},$$

then by combination of above representation together with (3.45) we get (3.46) as desired.

□

Note here that $J_{\alpha,p}(z)$ could also be written as follows

$$J_{\alpha,p}(z) = \frac{(z/2)^{\alpha}}{\Gamma(\alpha+1)}\left(1-\frac{z^2}{4}\sum_{n=0}^{\infty}\frac{\beta(n+1;\alpha+1;p)(-1)^n(z/2)^{2n}}{(2)_n n!}\right)$$

$$(Re(\alpha) > -1). \tag{3.47}$$

Corollary 3.3.2. *$J_{k,p}(z)$ is even for k even and odd for k odd.*

Proof. Set $\alpha = k$ in (3.46) and z to $(-z)$ to get

$$\begin{aligned}
J_{k,p}(-z) &= \frac{(-z)^k}{2^k\Gamma(k+1)} - \sum_{n=0}^{\infty}\frac{\beta(n+1;k+1;p)(-1)^n(-z)^{2n+k+2}}{\Gamma(k+1)2^{2n+k+2}(2)_n n!}\\
&= \frac{(-1)^k z^k}{2^k\Gamma(k+1)} - \sum_{n=0}^{\infty}\frac{\beta(n+1;k+1;p)(-1)^{n+k}z^{2n+k+2}}{\Gamma(k+1)2^{2n+k+2}(2)_n n!}\\
&= (-1)^k\cdot\left(\frac{z^k}{2^k\Gamma(k+1)} - \sum_{n=0}^{\infty}\frac{\beta(n+1;k+1;p)(-1)^n z^{2n+k+2}}{\Gamma(k+1)2^{2n+k+2}(2)_n n!}\right)\\
&= (-1)^k J_{k,p}(z)
\end{aligned} \tag{3.48}$$

Hence, for k even, $J_{k,p}(z)$ is even and for odd k's, $J_{k,p}(z)$ is odd. □

Corollary 3.3.3.

$$J_{0,p}(0) = 1. \tag{3.49}$$

Proof. Putting $\alpha = 0$ in (3.47) we get

$$J_{0,p}(z) = 1-\frac{z^2}{4}\sum_{n=0}^{\infty}\frac{\beta(n+1;\alpha+1;p)(-1)^n(z/2)^{2n}}{(2)_n n!}. \tag{3.50}$$

Setting $z = 0$ in (3.50) gives (3.49). □

Corollary 3.3.4.

$$J'_{1,p}(0) = \frac{1}{2}. \tag{3.51}$$

Proof. Since $\Gamma(2)=1$, then putting $\alpha=1$ in (3.46) gives

$$J_{1,p}(z)=\frac{z}{2}-\sum_{n=0}^{\infty}\frac{\beta(n+1;2;p)(-1)^n z^{2n+3}}{2^{2n+3}(2)_n n!} \tag{3.52}$$

and so,

$$J'_{1,p}(z)=\frac{1}{2}-\sum_{n=0}^{\infty}\frac{(2n+3)\beta(n+1;2;p)(-1)^n z^{2n+2}}{2^{2n+3}(2)_n n!}\,. \tag{3.53}$$

Hence, by setting $z=0$ in (3.53) we obtain (3.51). □

Corollary 3.3.5.

$$J'_{\alpha,p}(0)=0 \quad (Re(\alpha)>1). \tag{3.54}$$

Proof. Differentiating (3.46) with respect to z gives

$$J'_{\alpha,p}(z)=\frac{\alpha z^{\alpha-1}}{2^{\alpha}\Gamma(\alpha+1)}-\sum_{n=0}^{\infty}\frac{(2n+\alpha+2)\beta(n+1;\alpha+1;p)(-1)^n z^{2n+\alpha+1}}{\Gamma(\alpha+1)2^{2n+\alpha+2}(2)_n n!}\,. \tag{3.55}$$

Then, by setting $z=0$ in (3.55) we obtain (3.54) as $Re(\alpha)>1$. □

Theorem 3.3.6. *$J_{\alpha,p}(z)$ is uniformly convergent for $|z|<2$.*

Proof. From (3.46) we see that it is enough to consider the uniform convergence of this series

$$\sum_{n=0}^{\infty}\frac{\beta(n+1;\alpha+1;p)(-1)^n z^{2n+\alpha+2}}{\Gamma(\alpha+1)2^{2n+\alpha+2}(2)_n n!}\,. \tag{3.56}$$

By (A.3) we have

$$|\beta(n+1,\alpha+1,;p)|\le exp\,(-4p)|\beta(n+1,\alpha+1)| \tag{3.57}$$

and by (3.37) we have

$$|\beta(n+1,\alpha+1)|\le\left|\frac{1}{\alpha+1}\right|. \tag{3.58}$$

Then, for $|z| < 2$

$$
\begin{aligned}
\left|\sum_{n=0}^{\infty} \frac{\beta(n+1,\alpha+1;p)}{(2)_n}\frac{z^n}{n!}\right| &\leq \sum_{n=0}^{\infty}\left|\frac{\beta(n+1,\alpha+1;p)}{(2)_n}\frac{z^n}{n!}\right| \\
&< \sum_{n=0}^{\infty}\left|\frac{\beta(n+1;\alpha+1;p)}{(2)_n n!}\right| \\
&\leq e^{-4p}\sum_{n=0}^{\infty}\left|\frac{\beta(n+1,\alpha+1)}{(2)_n n!}\right| \\
&\leq \frac{e^{-4p}}{|\alpha+1|}\sum_{n=0}^{\infty}\left|\frac{1}{(2)_n n!}\right| \\
&= \frac{e^{-4p}}{|\alpha+1|}\sum_{n=0}^{\infty}\frac{1}{(2)_n n!} = \frac{e^{-4p}}{|\alpha+1|}\,{}_0F_1(-;2;1) \\
&= M(constant).
\end{aligned}
$$

Hence our result is proved by the Weierstrass M-test (see Theorem A.1 in the appendix). □

It is to be remarked here that the range of applicability of the uniform convergence is enlarged for the EBF if compared with that of ${}_0F_1(-;c;z;p)$. The following theorem, however, will provide an integral representation for the EBF.

Theorem 3.3.7.

$$
J_{\alpha,p}(z) = \frac{(z/2)^{\alpha}}{\Gamma(\alpha+1)}\left[1 - \frac{z}{2}\int_0^1 \frac{(1-t)^{\alpha}}{\sqrt{t}}\, exp\left[\frac{-p}{t(1-t)}\right] J_1(z\sqrt{t})dt\right]
$$

$$
(Re(\alpha) > -1). \tag{3.59}
$$

Proof. Set $c \to \alpha+1$ and $z \to -\frac{z^2}{4}$ in (3.38) to get

$$ {}_0F_1\left(-;\alpha+1;-\frac{z^2}{4};p\right) = 1-\frac{z^2}{4}\int_0^1 (1-t)^{\alpha}\, exp\left[\frac{-p}{t(1-t)}\right] {}_0F_1\left(-;2;-\frac{1}{4}(z\sqrt{t})^2\right)dt. \tag{3.60} $$

Now substituting (3.21) in (3.60) gives

$$ {}_0F_1\left(-;\alpha+1;-\frac{z^2}{4};p\right) = 1-\frac{z^2}{4}\int_0^1 (1-t)^{\alpha}\, exp\left[\frac{-p}{t(1-t)}\right] \frac{2}{z\sqrt{t}}J_1(z\sqrt{t})dt. \tag{3.61} $$

By putting (3.61) in (3.45) we obtain (3.59) as desired. □

It is worth mentioning here that the EBFs are also expressible in terms of the standard BF of order one.

Corollary 3.3.8.

$$ J_{\alpha,p}(z) = \frac{(z/2)^{\alpha}}{\Gamma(\alpha+1)}\left[1-\int_0^z \left(1-\frac{u^2}{z^2}\right)^{\alpha} exp\left[-\frac{pz^4}{u^2(z^2-u^2)}\right] J_1(u)du\right] $$

$$ (Re(\alpha)>-1). \tag{3.62} $$

Proof. Use the transformation $u = z\sqrt{t}$ in (3.59) to get the result. □

Notice that if we set $\alpha = 0$, we get

$$ J_{0,p}(z) = 1-\int_0^z exp\left[-\frac{pz^4}{u^2(z^2-u^2)}\right] J_1(u)du. \tag{3.63} $$

This of course gives $J_{0,p}(0) = 1$ as in Corollary 3.3.3. On the other hand, differentiating the integrand in (3.63) with respect to z will give an integral which is very complicated and so it is not a worthwhile elegant extension. Setting $p = 0$ in (3.63), we recover the standard result (3.24).

Corollary 3.3.9.

$$J_{\alpha,p}(z) = \frac{(z/2)^\alpha}{\Gamma(\alpha+1)} \left[1 - z\int_0^{\frac{\pi}{2}} (\cos\theta)^{2\alpha+1} exp\left(-4p\csc^2 2\theta\right)\right] J_1(z\sin\theta)d\theta$$

$$(Re(\alpha) > -1). \tag{3.64}$$

Proof. Use the transformation $u = z\sin\theta$ in (3.59) to get the result (3.64). □

Lemma 3.3.10.

$${}_0F_1(-;\alpha;z;p) = \frac{\Gamma(\alpha)}{i^{\alpha-1}z^{\frac{1}{2}(\alpha-1)}} J_{(\alpha-1),p}(2i\sqrt{z}) \quad (Re(\alpha) > 1). \tag{3.65}$$

Proof. Make the transformation $\alpha \to \alpha - 1$ and $z \to 2i\sqrt{z}$ in (3.45). Then, rearrange terms to get (3.65). □

Different bounds have been found for the standard BF. Examples include [37, eq (2.11-4)]:

$$|J_n(z)| \leq \frac{|z|^n}{2^n n!} exp\left(\frac{1}{4}|z|^2\right). \tag{3.66}$$

For details we refer to [37]. In the following theorem we give a bound for the EBF for α being an integer and z complex. Interestingly, the bound of $J_{n,p}(z)$ includes $J_n(z)$, the standard BF.

Theorem 3.3.11.

$$|J_{n,p}(z)| \leq \left|\frac{z^n}{2^{n-1}n!}\right| + e^{-4p}|J_n(z)|. \tag{3.67}$$

Proof. By setting $\alpha = n$ in (3.45) we get it to be

$$J_{n,p}(z) = \frac{z^n}{2^n n!} {}_0F_1\left(-;n+1;-\frac{z^2}{4}\right). \tag{3.68}$$

Taking the absolute value on both sides of (3.68) and using (3.42) with the fact that $e^{-4p} \leq 1$ for $p \geq 0$, we get

$$|J_{n,p}(z)| \leq \left|\frac{z^n}{2^n n!}\right| \left|2 + e^{-4p}\,{}_0F_1\left(-; n+1; -\frac{z^2}{4}\right)\right|$$
$$\leq \left|\frac{z^n}{2^{n-1} n!}\right| + e^{-4p} \left|\frac{z^n}{2^n n!}\,{}_0F_1\left(-; n+1; -\frac{z^2}{4}\right)\right|. \qquad (3.69)$$

But by (3.9) if we set $\alpha = n$, we have

$${}_0F_1(-; n+1; -\frac{z^2}{4}) = \frac{2^n n!}{z^n} J_n(z). \qquad (3.70)$$

Substitution of (3.70) in (3.69) gives (3.67). □

Corollary 3.3.12.

$$|J_{n,\infty}(z)| \leq \left|\frac{z^n}{2^{n-1} n!}\right|. \qquad (3.71)$$

Proof. Letting $p \to \infty$ in (3.67) gives (3.71). □

Corollary 3.3.13.

$$|J_{n,p}(z)| \leq \left|\frac{z^n}{2^n n!}\right| \left[2 + exp\left(-4p + \frac{1}{4}|z|^2\right)\right]. \qquad (3.72)$$

Proof. Substitution of (3.66) in (3.67) gives (3.72). □

Corollary 3.3.14.

$$J_{n,p}(0) = 0 \; \textit{for all} \;\; n > 0 \; \textit{and} \;\; p \geq 0. \qquad (3.73)$$

Proof. By (3.67), we have $\left|J_{n,p}(0)\right| \leq 0$. Hence, $J_{n,p}(0) = 0$. □

We conclude this section by providing the following results on the extended MBF where we omit the proofs because of their similarity to that of the EBF.

Theorem 3.3.15.

$$I_{\alpha,p}(z) = \frac{(z/2)^{\alpha}}{\Gamma(\alpha+1)}\left[1+\frac{z}{2}\int_0^1 \frac{(1-t)^{\alpha}}{\sqrt{t}}\, exp\left[\frac{-p}{t(1-t)}\right] I_1(z\sqrt{t})dt\right]$$

$$(Re(\alpha) > -1). \tag{3.74}$$

Again we see that in (3.74) all the EMBFs are expressible in terms of the standard modified Bessel function of first kind of order one.

Corollary 3.3.16.

$$I_{\alpha,p}(z) = \frac{(z/2)^{\alpha}}{\Gamma(\alpha+1)}\left[1+\int_0^z \left(1-\frac{u^2}{z^2}\right)^{\alpha} exp\left[-\frac{pz^4}{u^2(z^2-u^2)}\right] I_1(u)du\right]$$

$$(Re(\alpha) > -1). \tag{3.75}$$

We can see from this corollary that setting $\alpha = 0$, we get

$$I_{0,p}(z) = 1+\int_0^z exp\left[-\frac{pz^4}{u^2(z^2-u^2)}\right] I_1(u)du. \tag{3.76}$$

From here, we can easily see that differentiating with respect to z will give an integral which is very complicated. So, we conclude that the classical result given in (3.31) does not have a worthwhile elegant extension in the representation we are using.

Corollary 3.3.17.

$$I_{\alpha,p}(z) = \frac{(z/2)^{\alpha}}{\Gamma(\alpha+1)}\left[1+z\int_0^{\frac{\pi}{2}}(\cos\theta)^{2\alpha+1}exp\left(-4p\csc^2 2\theta\right)I_1(z\sin\theta)d\theta\right]$$

$$(Re(\alpha) > -1). \tag{3.77}$$

3.4 Incomplete CHF and Incomplete BF and their Extensions

The term "incomplete CHF" does not seem to exist in the literature. By incorporating the beta function in the series representation developed in this work for the CHF given in (3.16), we are able to define an *incomplete* CHF (ICHF) which is given as

$${}_0F_{1,t}(-;c;z) = 1+z\sum_{n=0}^{\infty}\frac{\beta_t(n+1,c)}{(2)_n}\frac{z^n}{n!} \quad (0 \le t \le 1). \tag{3.78}$$

If we use the integral representation of the incomplete beta function which is give by (2.3)

$$\beta_t(x,y) = \int_0^t h^{x-1}(1-h)^{y-1}dh \quad (Re(x) > 0, Re(y) > 0\, and\, 0 \le t \le 1),$$

then we can obtain the integral representation of the ICHF as

$$\begin{aligned}
{}_0F_{1,t}(-;c;z) &= 1+z\sum_{n=0}^{\infty}\frac{z^n}{(2)_n n!}\int_0^t h^n(1-h)^{c-1}dh \\
&= 1+z\int_0^t (1-h)^{c-1}\sum_{n=0}^{\infty}\frac{(hz)^n}{(2)_n n!}dh \\
&= 1+z\int_0^t (1-h)^{c-1}\,{}_0F_1(-;2;hz)dh. \qquad (3.79)
\end{aligned}$$

Similarly, by (3.33) we define the *extended* ICHF (EICHF) as

$$ {}_0F_{1,t}(-;c;z;p) = 1 + z\sum_{n=0}^{\infty} \frac{\beta_t(n+1,c;p)}{(2)_n}\frac{z^n}{n!} \quad (0 \le t \le 1) \tag{3.80} $$

which yields the integral representation as

$$ {}_0F_{1,t}(-;c;z;p) = 1 + z\int_0^t (1-h)^{c-1} exp\left[\frac{-p}{h(1-h)}\right] {}_0F_1(-;2;hz)dh. \tag{3.81} $$

There are different definitions of an "incomplete Bessel function" (IBF), [18, 23]. Here we provide yet another, by analogy with our ICHF:

$$ J_{\alpha,t}(z) = \frac{(z/2)^{\alpha}}{\Gamma(\alpha+1)} {}_0F_{1,t}\left(-;\alpha+1;-\frac{z^2}{4}\right). \tag{3.82} $$

Similarly, the *extended* IBF (EIBF) is defined as

$$ J_{\alpha,p,t}(z) = \frac{(z/2)^{\alpha}}{\Gamma(\alpha+1)} {}_0F_{1,t}\left(-;\alpha+1;-\frac{z^2}{4};p\right) \quad (Re(\alpha) > -1, 0 \le t \le 1). \tag{3.83} $$

By using the equation (3.78) to (3.80), we can easily obtain series representations for the IBF and EIBF respectively as

$$ J_{\alpha,t}(z) = \frac{(z/2)^{\alpha}}{\Gamma(\alpha+1)}\left(1 - \frac{z^2}{4}\sum_{n=0}^{\infty}\frac{\beta_t(n+1;\alpha+1)(-1)^n(z/2)^{2n}}{(2)_n n!}\right) $$

$$ (Re(\alpha) > -1, 0 \le t \le 1). \tag{3.84} $$

$$ J_{\alpha,p,t}(z) = \frac{(z/2)^{\alpha}}{\Gamma(\alpha+1)}\left(1 - \frac{z^2}{4}\sum_{n=0}^{\infty}\frac{\beta_t(n+1;\alpha+1;p)(-1)^n(z/2)^{2n}}{(2)_n n!}\right) $$

$$ (Re(\alpha) > -1, 0 \le t \le 1). \tag{3.85} $$

The integral representations are found from (3.79) and (3.81) to give

$$J_{\alpha,t}(z) = \frac{(z/2)^{\alpha}}{\Gamma(\alpha+1)}\left[1 - \frac{z}{2}\int_0^t \frac{(1-h)^{\alpha}}{\sqrt{h}} J_1(z\sqrt{h})dh\right] \tag{3.86}$$

and

$$J_{\alpha,p,t}(z) = \frac{(z/2)^{\alpha}}{\Gamma(\alpha+1)}\left[1 - \frac{z}{2}\int_0^t \frac{(1-h)^{\alpha}}{\sqrt{h}} exp\left[\frac{-p}{h(1-h)}\right] J_1(z\sqrt{h})dh\right]. \tag{3.87}$$

By using the transformation $u = z\sqrt{h}$, (3.86) and (3.87) can be respectively

$$J_{\alpha,t}(z) = \frac{(z/2)^{2}}{\Gamma(\alpha+1)}\left[1 - \int_0^{z\sqrt{t}} \left(1 - \frac{u^2}{z^2}\right)^{\alpha} J_1(u)du\right] \tag{3.88}$$

and

$$J_{\alpha,p,t}(z) = \frac{(z/2)^{\alpha}}{\Gamma(\alpha+1)}\left[1 - \int_0^{z\sqrt{t}} \left(1 - \frac{u^2}{z^2}\right)^{\alpha} exp\left[-\frac{pz^4}{u^2(z^2-u^2)}\right] J_1(u)du.\right]. \tag{3.89}$$

Remark. Most interestingly, we see from (3.88) and (3.89) that *the IBF and the EIBF we defined are again expressible in terms of the BF of order one.* For the special case of the standard BF we note that the BF of order α is expressible in terms of the standard BF of order 1.

Setting $\alpha = p = 0$ in (3.89) gives

$$J_{0,0,t}(z) = J_{0,t}(z) = 1 - \int_0^{z\sqrt{t}} J_1(u)du. \tag{3.90}$$

By differentiating both sides of (3.90) with respect to z we get the following elegant relation between the IBF and the BF as

$$J'_{0,t}(z) = J_1(z\sqrt{t}). \tag{3.91}$$

We can write the integral formula in alternative forms. Using the transformation $u = z\sin\theta$ in (3.88) and (3.89), then they both can respectively be written as

$$J_{\alpha,t}(z) = \frac{(z/2)^{\alpha}}{\Gamma(\alpha+1)}\left[1 - \int_{0}^{\sin^{-1}\sqrt{t}} (\cos\theta)^{2\alpha+1} J_1(z\sin\theta)du\right] \tag{3.92}$$

and

$$\begin{aligned} J_{\alpha,p,t}(z) =& \frac{(z/2)^{\alpha}}{\Gamma(\alpha+1)} \\ & \times \left[1 - z\int_{0}^{\sin^{-1}\sqrt{t}} (\cos\theta)^{2\alpha+1} exp\left(-4p\csc^{2}2\theta\right) J_1(z\sin\theta)d\theta\right]. \end{aligned} \tag{3.93}$$

The following two theorems give the integrals of the ICHF and the EIBF over the unit interval as a direct elegant relation with the complete version of both of them respectively.

Theorem 3.4.1.

$$\int_{0}^{1} {}_0F_{1,t}(-;\alpha;z;p)dt = {}_0F_1(-;\alpha+1;z;p). \tag{3.94}$$

Proof. By setting $c = \alpha$ in (3.78) and integrating both sides over the unit interval, we get

$$\int_{0}^{1} {}_0F_{1,t}(-;\alpha;z)dt = 1 + z\int_{0}^{1}\left(\sum_{n=0}^{\infty}\frac{\beta_t(n+1,\alpha)}{(2)_n}\frac{z^n}{n!}\right)dt \tag{3.95}$$

since the integral of the unity over the unit interval is again one. So, as before, interchanging the integral sign with the summation sign gives

$$\int_{0}^{1} {}_0F_{1,t}(-;\alpha;z)dt = 1 + z\sum_{n=0}^{\infty}\frac{z^n}{(2)_n n!}\int_{0}^{1}\beta_t(n+1,\alpha)\,dt. \tag{3.96}$$

Now, by using (2.11), the integral inside the summation sign on the right side becomes

$$\int_0^1 \beta_t(n+1,\alpha)\,dt = \beta(n+1,\alpha+1)$$

and so

$$\int_0^1 {}_0F_{1,t}(-;\alpha;z)dt = 1+z\sum_{n=0}^{\infty}\frac{z^n\beta(n+1,\alpha+1)}{(2)_n n!}. \tag{3.97}$$

By (3.33), the right side of (3.97) is exactly ${}_0F_1(-;\alpha+1;z;p)$ and hence (3.94) is obtained.

□

Theorem 3.4.2.

$$\int_0^1 J_{\alpha,p,t}(z)dt = \frac{2(\alpha+1)}{z}J_{\alpha+1,p}(z) \qquad (Re(\alpha)>-1). \tag{3.98}$$

Proof. Integrating both sides of (3.83) over the unit interval gives

$$\int_0^1 J_{\alpha,p,t}(z)dt = \frac{(z/2)^\alpha}{\Gamma(\alpha+1)}\int_0^1 {}_0F_{1,t}\left(-;\alpha+1;-\frac{z^2}{4};p\right)dt. \tag{3.99}$$

By (3.94), (3.99) is written as

$$\int_0^1 J_{\alpha,p,t}(z)dt = \frac{(z/2)^\alpha}{\Gamma(\alpha+1)}\,{}_0F_1\left(-;\alpha+2;-\frac{z^2}{4};p\right). \tag{3.100}$$

Now, by (3.45),

$${}_0F_1\left(-;\alpha+2;-\frac{z^2}{4};p\right) = \frac{\Gamma(\alpha+2)}{(z/2)^{\alpha+1}}J_{\alpha+1,p}(z). \tag{3.101}$$

Hence, substitution of (3.101) in (3.100) gives (3.98) as desired. □

These two results hold fully for the standard CHF and standard BF and does not seem to be present in the literature.

3.5 Graphical Representation of Extended (Bessel and Modified) Bessel Functions

Here we provide some graphs for the EBF and EMBF for some values of p and plot the functions for different values of α and vice versa. For that we use gpts points Gaussian rule adopted to improper integrals.

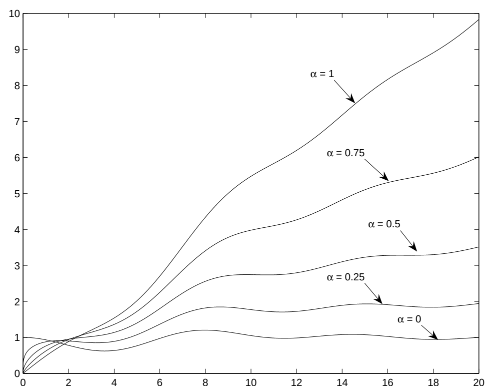

Figure 3.1: Plot of the EBF for $p = 0.2$ and different values of α.

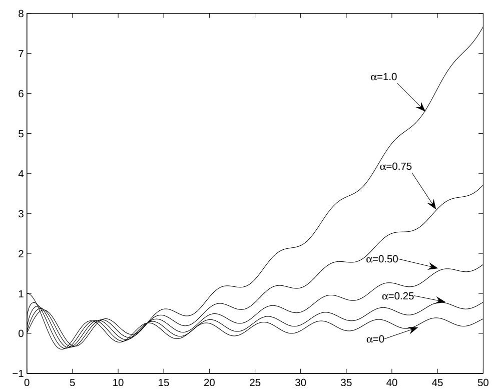

Figure 3.2: Plot of the EBF for $p = 0.0002$ and different values of α.

Remark: We can see that as p goes to zero, the EBF gets closer to the standard BF. However, as z increases, the integrand in (3.59) reduces and hence the function goes asymptotically as z^{α}.

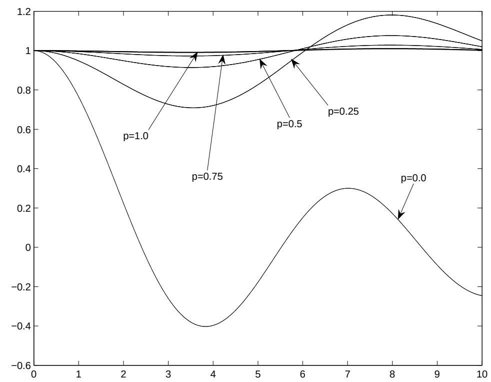

Figure 3.3: Plot of the EBF for $\alpha = 0$ and different values of p.

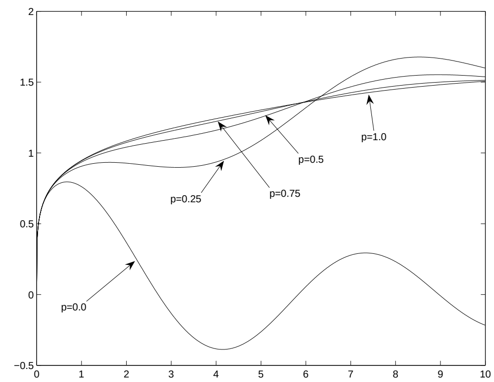

Figure 3.4: Plot of the EBF for $\alpha = 0.25$ and different values of p.

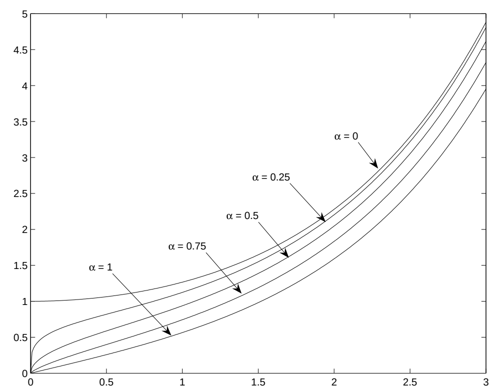

Figure 3.5: Plot of the EMBF for $p = 0$ and different values of α.

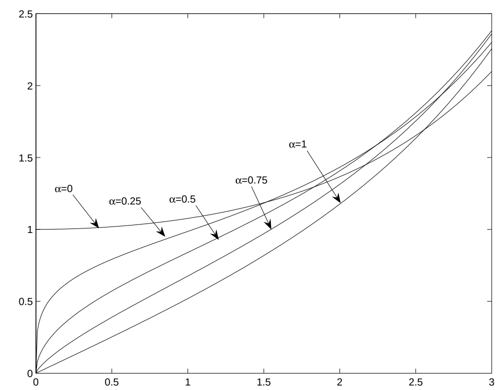

Figure 3.6: Plot of the EMBF for $p = 0.2$ and different values of α.

3.6 Concluding Remarks and Observations

The hypergeometric functions being a unified generalization of many of the special functions of mathematical physics, it was expected that any extension or generalization of these would lead to useful results for the known special functions and useful extensions of them [10]. Consequently, it seemed promising that an extension of the *confluent* version (CHF) be sought. In this chapter we introduced such an extension of the CHF and provided new integral representations for both the CHF and the ECHF. These led to *new integral representations for the standard Bessel function (BF) and its extension (EBF), such that both of them are expressed in terms of the standard BF of order one.* By following similar arguments, we provided parallel representations for a modified BF (MBF) and its extension (EMBF). Bounds, convergence and uniform convergence have been studied in detail for the ECHF and the EBF.

We went on to introduce an incomplete CHF (which does not seem to be available in the literature) and its extension. We also provided a new definition of an incomplete BF, by using the incomplete beta function, and its extension. This gave an elegant relation between the complete and incomplete versions through the integration of the incomplete one over the unit interval for both the functions. *The two results hold for the standard CHF and standard BF as well,* but do not seem to be available in the literature. This fact, that new insights can be obtained for the standard special functions by considering their generalizations, extensions and modifications, shows the power of this approach.

We also provided graphical representations of the functions discussed, with different choices of the parameter values. While a lot could be learned by looking at the details of the behavior of the functions, in the interest of space we limited our discussion of this representation.

Table 3.1: Some representative values of $J_{\alpha,p}(z)$ given by (3.59) with $p = 0.2$ and different values of α.

x	$\alpha = 0.00$	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 1.00$
0	1	0	0	0	0
0.1	0.9993	0.5214	0.2522	0.115	0.05
0.2	0.9971	0.6189	0.3561	0.1932	0.0999
0.3	0.9936	0.683	0.4351	0.2613	0.1495
0.4	0.9886	0.7309	0.5007	0.3232	0.1989
0.5	0.9824	0.7688	0.5573	0.3807	0.2478
0.6	0.9748	0.7995	0.6073	0.4345	0.2962
0.7	0.9659	0.8247	0.6518	0.4852	0.344
0.8	0.9559	0.8454	0.6919	0.5331	0.3911
0.9	0.9448	0.8624	0.728	0.5784	0.4375
1	0.9327	0.8761	0.7606	0.6213	0.483
1.1	0.9196	0.8869	0.7901	0.6619	0.5276
1.2	0.9057	0.8952	0.8167	0.7004	0.5713
1.3	0.8911	0.9012	0.8407	0.7368	0.6139
1.4	0.8759	0.9052	0.8623	0.7713	0.6556
1.5	0.8601	0.9075	0.8817	0.8039	0.6962
1.6	0.844	0.9081	0.899	0.8347	0.7358
1.7	0.8276	0.9074	0.9145	0.8638	0.7743
1.8	0.8111	0.9055	0.9283	0.8913	0.8118
1.9	0.7946	0.9026	0.9405	0.9173	0.8483
2	0.7782	0.8989	0.9514	0.942	0.8839
2.1	0.7621	0.8945	0.9612	0.9654	0.9185
2.2	0.7463	0.8897	0.9699	0.9877	0.9524
2.3	0.731	0.8845	0.9778	1.009	0.9854
2.4	0.7163	0.8793	0.985	1.0294	1.0177
2.5	0.7023	0.874	0.9918	1.0492	1.0495
2.6	0.6891	0.869	0.9982	1.0683	1.0808
2.7	0.6769	0.8642	1.0044	1.087	1.1117
2.8	0.6657	0.86	1.0106	1.1055	1.1423
2.9	0.6557	0.8564	1.017	1.1238	1.1728
3	0.6468	0.8536	1.0237	1.1422	1.2033

Table 3.2: Some representative values of $J_{\alpha,p}$ given by (3.59) with $p = 0.0002$ and different values of α.

x	$\alpha = 0.00$	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 1.00$
0	1	0	0	0	0
0.1	0.9975	0.5207	0.2519	0.1149	0.0499
0.2	0.9901	0.6155	0.3545	0.1924	0.0995
0.3	0.9777	0.6743	0.4305	0.2589	0.1483
0.4	0.9605	0.7145	0.4913	0.318	0.196
0.5	0.9387	0.7418	0.541	0.3712	0.2423
0.6	0.9123	0.7591	0.5817	0.4188	0.2867
0.7	0.8816	0.7679	0.6145	0.4613	0.3291
0.8	0.8468	0.7693	0.6402	0.4988	0.369
0.9	0.8082	0.7641	0.6591	0.5314	0.4061
1	0.766	0.7528	0.6718	0.559	0.4403
1.1	0.7206	0.7358	0.6785	0.5816	0.4712
1.2	0.6723	0.7137	0.6795	0.5993	0.4987
1.3	0.6214	0.6869	0.6751	0.6122	0.5225
1.4	0.5684	0.6556	0.6654	0.6202	0.5425
1.5	0.5136	0.6205	0.6509	0.6234	0.5587
1.6	0.4574	0.5818	0.6318	0.6219	0.5708
1.7	0.4001	0.54	0.6083	0.6158	0.5788
1.8	0.3424	0.4955	0.5808	0.6053	0.5828
1.9	0.2844	0.4488	0.5496	0.5905	0.5826
2	0.2267	0.4001	0.5151	0.5717	0.5784
2.1	0.1696	0.3501	0.4777	0.5491	0.5702
2.2	0.1136	0.299	0.4376	0.5229	0.5582
2.3	0.059	0.2474	0.3953	0.4934	0.5424
2.4	0.0062	0.1957	0.3511	0.4609	0.5231
2.5	-0.0445	0.1442	0.3056	0.4257	0.5003
2.6	-0.0928	0.0933	0.259	0.3882	0.4744
2.7	-0.1382	0.0436	0.2117	0.3487	0.4456
2.8	-0.1806	-0.0047	0.1643	0.3075	0.4142
2.9	-0.2197	-0.0512	0.117	0.265	0.3803
3	-0.2553	-0.0956	0.0703	0.2217	0.3445

Table 3.3: Some representative values of $J_{\alpha,p}$ given by (3.59) with $\alpha = 0.0$ and different values of p.

x	$p = 0.00$	$p = 0.25$	$p = 0.50$	$p = 0.75$	$p = 1.00$
0	1	1	1	1	1
0.1	0.9975	0.9994	0.9998	0.9999	1
0.2	0.99	0.9978	0.9993	0.9998	0.9999
0.3	0.9776	0.995	0.9985	0.9995	0.9998
0.4	0.9604	0.9912	0.9974	0.9992	0.9997
0.5	0.9385	0.9863	0.9959	0.9987	0.9996
0.6	0.912	0.9805	0.9941	0.9981	0.9994
0.7	0.8812	0.9736	0.9921	0.9975	0.9992
0.8	0.8463	0.9659	0.9898	0.9967	0.9989
0.9	0.8075	0.9573	0.9872	0.9959	0.9986
1	0.7652	0.9479	0.9844	0.995	0.9983
1.1	0.7196	0.9378	0.9814	0.994	0.998
1.2	0.6711	0.927	0.9781	0.993	0.9977
1.3	0.6201	0.9157	0.9747	0.9919	0.9973
1.4	0.5669	0.9039	0.9712	0.9908	0.997
1.5	0.5118	0.8918	0.9676	0.9896	0.9966
1.6	0.4554	0.8793	0.9638	0.9884	0.9962
1.7	0.398	0.8666	0.96	0.9872	0.9958
1.8	0.34	0.8539	0.9562	0.986	0.9954
1.9	0.2818	0.8411	0.9524	0.9848	0.995
2	0.2239	0.8284	0.9486	0.9836	0.9946
2.1	0.1666	0.8159	0.9449	0.9824	0.9942
2.2	0.1104	0.8037	0.9413	0.9812	0.9938
2.3	0.0555	0.7919	0.9377	0.9801	0.9934
2.4	0.0025	0.7806	0.9344	0.979	0.9931
2.5	-0.0484	0.7698	0.9312	0.978	0.9927
2.6	-0.0968	0.7597	0.9282	0.9771	0.9924
2.7	-0.1424	0.7503	0.9254	0.9762	0.9921
2.8	-0.185	0.7417	0.9229	0.9754	0.9919
2.9	-0.2243	0.734	0.9207	0.9747	0.9917
3	-0.2601	0.7272	0.9187	0.9741	0.9915

Table 3.4: Some representative values of $J_{\alpha,p}$ given by (3.59) with $\alpha = 0.25$ and different values of p.

x	$p = 0.00$	$p = 0.25$	$p = 0.50$	$p = 0.75$	$p = 1.00$
0	0	0	0	0	0
0.1	0.5207	0.5215	0.5216	0.5217	0.5217
0.2	0.6155	0.6193	0.6201	0.6203	0.6204
0.3	0.6743	0.6838	0.6857	0.6863	0.6865
0.4	0.7144	0.7324	0.7362	0.7373	0.7376
0.5	0.7417	0.7713	0.7775	0.7793	0.7798
0.6	0.7589	0.8033	0.8125	0.8152	0.8161
0.7	0.7677	0.8301	0.843	0.8468	0.848
0.8	0.769	0.8526	0.8699	0.875	0.8766
0.9	0.7637	0.8717	0.894	0.9005	0.9026
1	0.7522	0.8877	0.9157	0.9239	0.9265
1.1	0.7352	0.9011	0.9353	0.9454	0.9485
1.2	0.7129	0.9122	0.9533	0.9653	0.9691
1.3	0.6859	0.9213	0.9698	0.9839	0.9884
1.4	0.6545	0.9286	0.9849	1.0014	1.0066
1.5	0.6192	0.9343	0.9989	1.0178	1.0238
1.6	0.5804	0.9386	1.0119	1.0333	1.0401
1.7	0.5384	0.9416	1.024	1.048	1.0556
1.8	0.4937	0.9436	1.0352	1.062	1.0704
1.9	0.4467	0.9447	1.0458	1.0753	1.0846
2	0.3978	0.945	1.0558	1.0881	1.0982
2.1	0.3475	0.9447	1.0652	1.1003	1.1114
2.2	0.2962	0.944	1.0742	1.1121	1.124
2.3	0.2444	0.9429	1.0827	1.1234	1.1362
2.4	0.1923	0.9417	1.091	1.1344	1.148
2.5	0.1406	0.9404	1.0989	1.145	1.1594
2.6	0.0895	0.9391	1.1067	1.1553	1.1705
2.7	0.0394	0.9381	1.1143	1.1653	1.1813
2.8	-0.0092	0.9374	1.1219	1.1752	1.1919
2.9	-0.056	0.9372	1.1293	1.1848	1.2021
3	-0.1006	0.9375	1.1368	1.1942	1.2121

Table 3.5: Some representative values of $I_{\alpha,p}$ given by (3.74) with $p=0$ and different values of α.

x	$p=0.00$	$p=0.25$	$p=0.50$	$p=0.75$	$p=1.00$
0	1	0	0	0	0
0.1	1.0025	0.5227	0.2527	0.1152	0.0501
0.2	1.01	0.6254	0.3592	0.1946	0.1005
0.3	1.0226	0.699	0.4436	0.2656	0.1517
0.4	1.0404	0.7616	0.5182	0.3329	0.204
0.5	1.0635	0.8197	0.588	0.3986	0.2579
0.6	1.092	0.8765	0.6558	0.4641	0.3137
0.7	1.1263	0.934	0.7234	0.5306	0.3719
0.8	1.1665	0.9938	0.7922	0.5988	0.4329
0.9	1.213	1.0567	0.8633	0.6696	0.4971
1	1.2661	1.1239	0.9377	0.7437	0.5652
1.1	1.3262	1.196	1.0161	0.8218	0.6375
1.2	1.3937	1.2739	1.0994	0.9047	0.7147
1.3	1.4693	1.3583	1.1885	0.993	0.7973
1.4	1.5534	1.45	1.2841	1.0875	0.8861
1.5	1.6467	1.5499	1.3872	1.1891	0.9817
1.6	1.75	1.6588	1.4985	1.2985	1.0848
1.7	1.864	1.7775	1.619	1.4166	1.1963
1.8	1.9896	1.9072	1.7497	1.5444	1.3172
1.9	2.1277	2.0487	1.8918	1.683	1.4482
2	2.2796	2.2034	2.0462	1.8335	1.5906
2.1	2.4463	2.3723	2.2144	1.997	1.7455
2.2	2.6291	2.5569	2.3976	2.1749	1.9141
2.3	2.8296	2.7586	2.5974	2.3687	2.0978
2.4	3.0493	2.9791	2.8153	2.5799	2.2981
2.5	3.2898	3.2201	3.0531	2.8103	2.5167
2.6	3.5533	3.4836	3.3127	3.0617	2.7554
2.7	3.8417	3.7717	3.5963	3.3361	3.0161
2.8	4.1573	4.0867	3.9061	3.6359	3.3011
2.9	4.5027	4.4311	4.2447	3.9635	3.6126
3	4.8808	4.8078	4.6148	4.3216	3.9534

Table 3.6: Some representative values of $I_{\alpha,p}$ given by (3.74) with $p = 0.2$ and different values of α.

x	$p = 0.00$	$p = 0.25$	$p = 0.50$	$p = 0.75$	$p = 1.00$
0	1	0	0	0	0
0.1	1.0007	0.522	0.2524	0.1151	0.05
0.2	1.0029	0.6219	0.3575	0.1938	0.1001
0.3	1.0065	0.6903	0.439	0.2632	0.1505
0.4	1.0116	0.7448	0.5087	0.3276	0.2012
0.5	1.0182	0.7918	0.5713	0.3888	0.2523
0.6	1.0264	0.8343	0.6293	0.4478	0.3039
0.7	1.0362	0.8739	0.6842	0.5055	0.3563
0.8	1.0478	0.9119	0.7371	0.5624	0.4095
0.9	1.0611	0.949	0.7887	0.619	0.4636
1	1.0763	0.9859	0.8396	0.6756	0.5189
1.1	1.0935	1.0231	0.8904	0.7325	0.5754
1.2	1.1129	1.061	0.9415	0.7901	0.6334
1.3	1.1345	1.1	0.9933	0.8487	0.693
1.4	1.1586	1.1404	1.0462	0.9086	0.7544
1.5	1.1853	1.1825	1.1005	0.9701	0.8179
1.6	1.2149	1.2269	1.1566	1.0335	0.8837
1.7	1.2474	1.2736	1.215	1.0992	0.9521
1.8	1.2833	1.3232	1.2759	1.1675	1.0233
1.9	1.3227	1.376	1.3397	1.2387	1.0977
2	1.366	1.4324	1.4069	1.3133	1.1757
2.1	1.4134	1.4927	1.478	1.3918	1.2576
2.2	1.4653	1.5575	1.5534	1.4745	1.3438
2.3	1.5222	1.6272	1.6335	1.5619	1.4348
2.4	1.5844	1.7023	1.7191	1.6547	1.5312
2.5	1.6524	1.7833	1.8106	1.7534	1.6334
2.6	1.7267	1.8709	1.9087	1.8586	1.7421
2.7	1.8078	1.9658	2.014	1.9711	1.858
2.8	1.8964	2.0686	2.1275	2.0916	1.9818
2.9	1.9932	2.18	2.2498	2.221	2.1144
3	2.0987	2.301	2.382	2.3601	2.2566

Chapter 4

Lipschitz and Hankel's Integrals for $J_{\alpha,p}$ the Extended Bessel Function of the First Kind

The question that whether an extension of a function is a good extension or not depends on several factors. Preserving the properties of the original function is one factor. The other factor is the elegancy of the generalized properties. Of course, the next important point would be the application part of that function and to what extent it will emerge there.

Bessel functions play an important role in many engineering areas as in potential theory and telecommunications. In potential theory, for example, $e^{-\rho t}J_0(zt)$ is taken as the potential function which if integrated from zero to infinity turns to be the Lipschitz integral [37] as we will see. In the previous chapter on the EBF, we made the extension naturally from that of the extension of the beta function [9, 11] and we noticed that some properties are preserved. So, as a continuation to the research work we did in the area of extending the standard BF to the EBF, we try in this chapter, to develop some of the existent properties and seek their extension. We are actually working some more applied properties on this newly extended function and we seek further research where such function finds its way to be applied in the other science areas.

This chapter consists of eight sections. In the following section we introduce and define the generalized extended hypergeometric function. Then, in section three, Lipschitz integral is presented where its generalization for the EBF is found. As Hankel's integral is the generalization of Lipschitz integral, we find the Hankel's integral for the EBF in section four. Then it comes the application section. In that section we apply our generalizations to study two special cases where we cover *the Laplace transform for the EBF of zero order.* After that, in section six, we study the asymptotic behavior of the EBF and in section seven, we find the Mellin-Barnes integral. The chapter is finalized with some concluding remarks.

4.1 The Generalized Extended Hypergeometric Function

Some properties of the EGHF introduced in (2.5) have been investigated by Chaudhry et. al. [10]. Although the generalized GHF is known in the literature [2, 16], the generalized extended GHF does not seem to be so. Here, in this work we will just introduce such a function for a purpose that would appear later and we leave a more extensive study of it for further research work.

We define the *generalized extended* Gauss hypergeomteric function (GEGHF) as follows:

$$F_p\left(\begin{matrix} a \quad\quad d \\ e \end{matrix}\; ;b;c;z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{n=0}^{\infty}\frac{(a)_n(d)_n\beta(b+n,c-b;p)z^n}{(e)_n n!}. \tag{4.1}$$

In the following two theorems we prove its convergence for $|z| < 1$ and its satisfaction to the differential formula exactly as the standard GHF [2] and the EGHF [10]. However, before that we state the following property about the Pochhammer symbol defined in (3.12) as

$$(a)_n = a \cdot (a+1) \cdot (a+2) \cdots (a+n-1) \quad (n = 1,2,...,),$$

where $(a)_0 = 1$.

Lemma 4.1.1.

$$(a)_{n+1} = a(a+1)_n \tag{4.2}$$

$$= (a+n)(a)_n \tag{4.3}$$

Proof.

$$(a)_{n+1} = a \cdot [(a+1) \cdot (a+2) \cdots (a+n-1)(a+n)] = a(a+1)_n$$
$$= [a \cdot (a+1) \cdot (a+2) \cdots (a+n-1)](a+n) = (a+n)(a)_n$$

as desired. □

Theorem 4.1.2. $F_p\left(\begin{matrix} a & & d \\ & e & \end{matrix};b;c;z\right)$ *is convergent for* $|z| < 1$.

Proof. By noting that $(a)_{n+1} = (a+n)(a)_n$ and by using the ratio test, we have

$$\lim_{n\to\infty}\left|\frac{(a)_{n+1}(d)_{n+1}\beta(n+b+1,c-b;p)z^{n+1}}{(e)_{n+1}(n+1)!}\cdot\frac{(e)_n n!}{(a)_n(d)_n\beta(n+b,c-b;p)z^n}\right|$$

$$= |z|\lim_{n\to\infty}\left|\frac{(a+n)(d+n)}{(e+n)(n+1)}\cdot\frac{\beta(n+b+1,c-b;p)}{\beta(n+b,c-b;p)}\right|. \tag{4.4}$$

By (3.34), (4.4) equals $|z|$ and hence we have the convergence for $|z| < 1$. □

Theorem 4.1.3. $F_p\left(\begin{array}{ccc} a & & d \\ & e & \end{array};b;c;z\right)$ *satisfies the differential formula which is given by*

$$\frac{d^k}{dz^k}F_p\left(\begin{array}{ccc} a & & d \\ & e & \end{array};b;c;z\right)$$

$$=\frac{(a)_k(b)_k(d)_k}{(c)_k(e)_k}F_p\left(\begin{array}{ccc} a+k & & d+k \\ & e+k & \end{array};b+k;c+k;z\right). \tag{4.5}$$

Proof. We prove that by induction. For $k=1$, we have

$$\frac{d}{dz}F_p\left(\begin{array}{ccc} a & & d \\ & e & \end{array};b;c;z\right)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{n=1}^{\infty}\frac{(a)_n(d)_n\beta(b+n,c-b;p)z^{n-1}}{(e)_n(n-1)!}. \tag{4.6}$$

Shifting the index $n\to n+1$, (4.6) is written as

$$\frac{d}{dz}F_p\left(\begin{array}{ccc} a & & d \\ & e & \end{array};b;c;z\right)$$

$$=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b))}\sum_{n=0}^{\infty}\frac{(a)_{n+1}(d)_{n+1}\beta(b+n+1,c-b;p)z^n}{(e)_{n+1}n!}. \tag{4.7}$$

Now, since $(c+1)-(b+1)=c-b$, then by using (4.2) and the property of the Gamma function

$$\Gamma(\alpha+1)=\alpha\Gamma(\alpha), \tag{4.8}$$

(4.7) could be written as

$$\frac{d}{dz}F_p\left(\begin{array}{ccc} a & & d \\ & & & ;b;c;z \\ & e & \end{array}\right)$$

$$\begin{aligned}
&=\frac{b\Gamma(c+1)}{c\Gamma(b+1)\Gamma(c-b))}\sum_{n=0}^{\infty}\frac{a(a+1)_n d(d+1)_n \beta(b+n+1,c-b;p)z^n}{e(e+1)_n n!}\\
&=\frac{abd}{ce}\frac{\Gamma(c+1)}{\Gamma(b+1)\Gamma(c-b))}\sum_{n=0}^{\infty}\frac{(a+1)_n(d+1)_n \beta(b+n+1,c-b;p)z^n}{(e+1)_n n!}\\
&=\frac{abd}{ce}F_p\left(\begin{array}{ccc} a+1 & & d+1 \\ & & & ;b+1;c+1;z \\ & e+1 & \end{array}\right)
\end{aligned} \tag{4.9}$$

as $(a)_1 = a$ and so the statement is true for $k=1$. Now we assume that (4.5) is true for $k=m$. That is

$$\frac{d^m}{dz^m}F_p\left(\begin{array}{ccc} a & & d \\ & & & ;b;c;z \\ & e & \end{array}\right)$$

$$=\frac{(a)_m(b)_m(d)_m}{(c)_m(e)_m}F_p\left(\begin{array}{ccc} a+m & & d+m \\ & & & ;b+m;c+m;z \\ & e+m & \end{array}\right). \tag{4.10}$$

We then prove that

$$\begin{aligned}
\frac{d^{m+1}}{dz^{m+1}}F_p\left(\begin{array}{ccc} a & & d \\ & & & ;b;c;z \\ & e & \end{array}\right) &= \frac{(a)_{m+1}(b)_{m+1}(d)_{m+1}}{(c)_{m+1}(e)_{m+1}}\\
&\times F_p\left(\begin{array}{ccc} a+m+1 & & d+m+1 \\ & & & ;b+m+1;c+m+1;z \\ & e+m+1 & \end{array}\right).
\end{aligned} \tag{4.11}$$

is true. To do so, we have by (4.10)

$$\frac{d^{m+1}}{dz^{m+1}}F_p\left(\begin{array}{ccc} a & & d \\ & e & \end{array};b;c;z\right)=\frac{d}{dz}\left[\frac{d^m}{dz^m}F_p\left(\begin{array}{ccc} a & & d \\ & e & \end{array};b;c;z\right)\right]$$

$$=\frac{d}{dz}\left[\frac{(a)_m(b)_m(d)_m}{(c)_m(e)_m}F_p\left(\begin{array}{ccc} a+m & & d+m \\ & e+m & \end{array};b+m;c+m;z\right)\right]$$

$$=\frac{(a)_m(b)_m(d)_m}{(c)_m(e)_m}\frac{\Gamma(c+m)}{\Gamma(b+m)\Gamma(c-b)}$$

$$\times\sum_{n=1}^{\infty}\frac{(a+m)_n(d+m)_n\beta(b+n+m,c-b;p)z^{n-1}}{(e+m)_n(n-1)!}. \tag{4.12}$$

(Shifing the summation index one step up give it as)

$$=\frac{(a)_m(b)_m(d)_m}{(c)_m(e)_m}\frac{\Gamma(c+m)}{\Gamma(b+m)\Gamma(c-b)}$$

$$\times\sum_{n=0}^{\infty}\frac{(a+m)_{n+1}(d+m)_{n+1}\beta(b+n+m+1,c-b;p)z^{n}}{(e+m)_{n+1}(n)!}. \tag{4.13}$$

Now by using (4.2) and (4.8), (4.13) could be written as

$$=\frac{(a+m)(a)_m(b+m)(b)_m(d+m)(d)_m}{(c+m)(c)_m(e+m)(e)_m}\frac{\Gamma(c+m+1)}{\Gamma(b+m+1)\Gamma(c-b)}$$

$$\times\sum_{n=0}^{\infty}\frac{(a+m+1)_n(d+m+1)_n\beta(b+n+m+1,c-b;p)z^{n}}{(e+m+1)_n(n)!}. \tag{4.14}$$

By (4.3), (4.14) becomes

$$=\frac{(a)_{m+1}(b)_{m+1}(d)_{m+1}}{(c)_{m+1}(e)_{m+1}}\frac{\Gamma(c+m+1)}{\Gamma(b+m+1)\Gamma(c-b)}$$

$$\times\sum_{n=0}^{\infty}\frac{(a+m+1)_n(d+m+1)_n\beta(b+n+m+1,c-b;p)z^{n}}{(e+m+1)_n(n)!} \tag{4.15}$$

which is (4.11) and hence the induction proof is complete. □

4.2 Lipschitz Integral for the EBF

It has been shown by Lipschitz [37] that

$$\int_0^\infty e^{-at} J_0(ht)dt = \frac{1}{\sqrt{a^2+h^2}} \quad (Re(a) > 0). \tag{4.16}$$

Although proof of (4.16) is available in the literature [37], we provide in the appendix another proof using our new representation of the BF given by(3.19). Here in in this section, however, we generalize the the above result to the EBF as in the following theorem.

Theorem 4.2.1.

$$\int_0^\infty e^{-az} J_{0,p}(hz)dz = \frac{1}{a} - \frac{h^2}{2a^3} F_p\left(\frac{3}{2}, 1, 2, -\frac{h^2}{a^2}\right) \tag{4.17}$$

Proof. As $J_{\alpha,p}(z)$ is by (3.59) written as :

$$J_{\alpha,p}(z) = \frac{z^\alpha}{2^\alpha \Gamma(\alpha+1)} - \frac{z^{\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+1)} \int_0^1 \frac{(1-t)^\alpha}{\sqrt{t}} exp\left[\frac{-p}{t(1-t)}\right] J_1(z\sqrt{t})dt,$$

then

$$\begin{aligned} &\int_0^\infty e^{-az} J_{0,p}(hz)dz \\ =&\int_0^\infty e^{-az}dz - \int_0^1 \frac{h}{2\sqrt{t}} exp\left[\frac{-p}{t(1-t)}\right] \left(\int_0^\infty e^{-az} z J_1(hz\sqrt{t})dz\right) dt \\ =&\frac{1}{a} - \int_0^1 \frac{h}{2\sqrt{t}} exp\left[\frac{-p}{t(1-t)}\right] \left(\int_0^\infty z e^{-az} J_1(hz\sqrt{t})dz\right) dt. \end{aligned} \tag{4.18}$$

But by (A.16)

$$\int_0^\infty z e^{-az} J_1(hz\sqrt{t})dz = \frac{(2h\sqrt{t})\Gamma\left(\frac{3}{2}\right)}{\sqrt{\pi}(a^2+h^2t)^{\frac{3}{2}}} = \frac{h\sqrt{t}}{(a^2+h^2t)^{\frac{3}{2}}}. \tag{4.19}$$

Now, putting (4.19) in (4.18) gives

$$\int_0^\infty e^{-az} J_{0,p}(hz)dz = \frac{1}{a} - \int_0^1 \frac{h^2}{2(a^2+h^2t)^{\frac{3}{2}}} exp\left[\frac{-p}{t(1-t)}\right] dt. \tag{4.20}$$

For $p = 0$, we are back as in previous case where the integral on the right side is exactly (A.18) which could be evaluated by using a simple transformation to give (4.16). For the details, see the proof of Theorem (A.0.5) in the appendix . For $p \neq 0$, the integral on the right side of (4.20) is written as:

$$\int_0^1 \frac{1}{(a^2+h^2t)^{\frac{3}{2}}} exp\left[\frac{-p}{t(1-t)}\right] dt = \frac{1}{a^3} F_p\left(\frac{3}{2}, 1, 2, -\frac{h^2}{a^2}\right) \tag{4.21}$$

and hence,

$$\int_0^\infty e^{-az} J_{0,p}(hz)dz = \frac{1}{a} - \frac{h^2}{2a^3} F_p\left(\frac{3}{2}, 1, 2, -\frac{h^2}{a^2}\right) \tag{4.22}$$

as desired. □

Remark 1. Up to the stage of finding (4.20) in the above proof, it seems that the right side of (4.20) is not integrable in a closed mathematical form that incorporates the standard known special functions. If we also check with some of the available computer software that performs symbolic integration like Maple or Mathematica, they also fail to give a closed form of integration. Hence, it appears that the extension of Lipschitz integral is just as it is in (4.20).

However, if we go on and give our generalization to Hankel's integral as we will see in the next section, we will see that a closed form of (4.20) is possible. The answer as expected is not given in terms of classical (standard) known special functions. It is given in terms of the newly extended hypergeometric function introduced by Chaudhry et. al. [10] that we mentioned before. Here, in the above proof, we just gave the result (4.22) and the complete proof of that will be provided later.

The following corollary gives the asymptotic behavior of the generalized Lipschitz integral.

Corollary 4.2.2.

$$\int_0^\infty e^{-az} J_{0,p}(hz)dz \sim \frac{1}{a} \quad (as\ p \to \infty). \tag{4.23}$$

Proof. In (4.20) when $p \to \infty$, $exp\left[\frac{-p}{t(1-t)}\right] \to 0$ and so (4.23) is obtained. □

4.3 Hankel's Integral for the EBF

The known Hankel's generalization of Lipschitz integral [37] is given by

$$\int_0^\infty e^{-az} z^{\mu-1} J_\alpha(hz)dz$$

$$= \frac{\left(\frac{1}{2}h/a\right)^\alpha \Gamma(\mu+\alpha)}{a^\mu \Gamma(\alpha+1)} {}_2F_1\left(\frac{\mu+\alpha}{2}; \frac{\mu+\alpha+1}{2}; \alpha+1; -\frac{h^2}{a^2}\right) (|h| < |\alpha|). \tag{4.24}$$

Here we want to see how such a result being generalized under the extension of the standard BF.

Theorem 4.3.1.

$$\int_0^\infty e^{-az} z^{\mu-1} J_{\alpha,p}(hz)dz = \frac{h^\alpha \Gamma(\mu+\alpha)}{a^{\alpha+\mu} 2^\alpha \Gamma(\alpha+1)}$$

$$\times \left[1 - \frac{h^2(\mu+\alpha+1)(\mu+\alpha)}{4a^2(\alpha+1)} F_p\left(\begin{array}{ccc}\frac{\mu+\alpha+2}{2} & & \frac{\mu+\alpha+3}{2} \\ & 2 & \end{array}; 1; \alpha+2; -\frac{h^2}{a^2}\right)\right]. \tag{4.25}$$

Proof. By putting (3.59) in the left side of (4.25) we get:

$$\int_0^\infty e^{-az} z^{\mu-1} J_{\alpha,p}(hz)dz = \int_0^\infty \frac{z^{\alpha+\mu-1}h^\alpha}{2^\alpha \Gamma(a+1)} e^{-az} dz$$

$$-\int_0^1 \frac{h^{\alpha+1}(1-t)^\alpha}{2^{\alpha+1}\Gamma(\alpha+1)\sqrt{t}} exp\left[\frac{-p}{t(1-t)}\right] \left(\int_0^\infty e^{-az} z^{\alpha+\mu} J_1(hz\sqrt{t})dz\right) dt. \tag{4.26}$$

If we use (A.11)

$$\int_0^\infty e^{-az} z^{x-1} dz = a^{-x}\Gamma(x)$$

on the first term of (4.25) (right side) and apply (4.24) on the integral inside the parentheses, we obtain:

$$\int_0^\infty e^{-az} z^{\mu-1} J_{\alpha,p}(hz)dz = \frac{h^\alpha a^{-\alpha-\mu}}{2^\alpha \Gamma(\alpha+1)} \Gamma(\alpha+\mu)$$

$$-\int_0^1 \frac{h^{\alpha+2}(1-t)^\alpha \Gamma(\alpha+\mu+2)}{2^{\alpha+2} a^{\alpha+\mu+2} \Gamma(\alpha+1)} exp\left[\frac{-p}{t(1-t)}\right] {}_2F_1\left(\frac{\alpha+\mu+2}{2}; \frac{\alpha+\mu+3}{2}; 2; -\frac{h^2 t}{a^2}\right) dt. \tag{4.27}$$

Now, if we use the following series representation of ${}_2F_1$

$${}_2F_1\left(\frac{\alpha+\mu+2}{2}; \frac{\alpha+\mu+3}{2}; 2; -\frac{h^2 t}{a^2}\right)$$

$$= \sum_{n=0}^\infty \frac{[(\alpha+\mu+2)/2]_n [(\alpha+\mu+3)/2]_n (-1)^n h^{2n} t^n}{[2]_n a^{2n} n!} \tag{4.28}$$

in the (right side) integral of in (4.27), we get it equal to

$$\int_0^1 \frac{h^{\alpha+2}(1-t)^\alpha \Gamma(\alpha+\mu+2)}{2^{\alpha+2} \alpha^{\alpha+\mu+2} \Gamma(\alpha+1)} exp\left[\frac{-p}{t(1-t)}\right]$$

$$\times \sum_{n=0}^\infty \frac{[(\alpha+\mu+2)/2]_n [(\alpha+\mu+3)/2]_n (-1)^n h^{2n} t^n}{[2]_n a^{2n} n!} dt. \tag{4.29}$$

Interchanging the integral sign with the summation sign and collecting related terms we get it equal to

$$\sum_{n=0}^{\infty}\frac{[(\alpha+\mu+2)/2]_n[(\alpha+\mu+3)/2]_n\Gamma(\alpha+\mu+2)(-1)^n h^{2n+\alpha+2}}{[2]_n 2^{\alpha+2}a^{2n+\alpha+\mu+2}\Gamma(\alpha+1)n!}$$

$$\times\int_0^1(1-t)^{\alpha}t^n\,exp\left[\frac{-p}{t(1-t)}\right]dt. \tag{4.30}$$

Note that the integral inside the above summation is the extended beta function $\beta(n+1,\alpha+1;p)$. Hence

$$\int_0^{\infty}e^{-az}z^{\mu-1}J_{\alpha,p}(hz)dz$$

$$=\frac{h^{\alpha}a^{-\alpha-\mu}}{2^{\alpha}\Gamma(\alpha+1)}\Gamma(\alpha+\mu)-\sum_{n=0}^{\infty}\left(\frac{[(\alpha+\mu+2)/2]_n[(\alpha+\mu+3)/2]_n}{[2]_n}\right)\times$$

$$\left(\frac{(-1)^n h^{2n+\alpha+2}\Gamma(\alpha+\mu+2)\beta(n+1,\alpha+1;p)}{2^{\alpha+2}a^{2n+\alpha+\mu+2}\Gamma(\alpha+1)n!}\right) \tag{4.31}$$

By noting that the series term of (4.31) is the newly defined GEGHF we introduced in section one, (4.31) can be reduced to (4.25) as desired. □

Remark 2. (4.25) is our generalization of Hankel's integral for the EBF. By setting $(p=0)$ in above result, we are back to (4.24). See the appendix for the details. In the letter section, however, we will see some of the importance of (4.25).

However, before we end this section, we present the following corollary which comes as a consequence of what we have developed so far.

Corollary 4.3.2.

$$ {}_2F_1\left(\frac{1}{2};1;1;-\frac{b^2}{a^2}\right) = {}_1F_0\left(\frac{1}{2};-;-;-\frac{b^2}{a^2}\right) = \frac{a}{\sqrt{a^2+b^2}}. \tag{4.32} $$

Proof. In the proof of Theorem (A.0.5) in the appendix we get the following integral

$$ \int_0^\infty e^{-az} z^{2\alpha+1} J_1(bz\sqrt{t})dz \tag{4.33} $$

in (A.14) which we did not evaluate to ease the process. Now, we evaluate the above integral by using (4.24) which we rewrite as

$$ \int_0^\infty e^{-az} z^{\mu-1} J_\nu(hz)dz $$

$$ = \frac{\left(\frac{1}{2}h/a\right)^\nu \Gamma(\mu+\nu)}{a^\mu \Gamma(\nu+1)} {}_2F_1\left(\frac{\mu+\nu}{2};\frac{\mu+\nu+1}{2};\nu+1;-\frac{h^2}{a^2}\right) \tag{4.34} $$

to get

$$ \int_0^\infty e^{-az} z^{2\alpha+1} J_1(bz\sqrt{t})dz = \frac{b\sqrt{t}\Gamma(2\alpha+3)}{2a^{2\alpha+3}} {}_2F_1\left(\frac{2\alpha+3}{2};\alpha+2;2;-\frac{b^2t}{a^2}\right) \tag{4.35} $$

where we set $\nu = 1$ and $\mu = 2\alpha+2$ in (4.34). Putting (4.35) back in (A.14) we get it as

$$ \begin{aligned} \int_0^\infty e^{-az} z^\alpha J_\alpha(bz)dz &= \frac{b^\alpha}{2^\alpha \Gamma(\alpha+1)} a^{-2\alpha-1} \Gamma(2\alpha+1) \\ &- \int_0^1 \frac{b^{\alpha+1}(1-t)^\alpha}{2^{\alpha+1}\Gamma(\alpha+1)\sqrt{t}} \left[\frac{b\sqrt{t}\Gamma(2\alpha+3)}{2a^{2\alpha+3}} {}_2F_1\left(\frac{2\alpha+3}{2};\alpha+2;2;-\frac{b^2t}{a^2}\right)\right] dt. \end{aligned} \tag{4.36} $$

Setting $\alpha = 0$ in the above equation gives

$$ \int_0^\infty e^{-az} J_0(bz)dz = \frac{1}{a} - \int_0^1 \frac{b^2}{2a^3} {}_2F_1\left(\frac{3}{2};2;2;-\frac{b^2t}{a^2}\right) dt. \tag{4.37} $$

as $\Gamma(3)=2$. Now, comparing (4.37) with (A.18) and (A.19) we obtain

$$\int_0^1 \frac{b^2}{2a^3}\,{}_2F_1\left(\frac{3}{2};2;2;-\frac{b^2t}{a^2}\right)dt = \int_0^1 \frac{b^2}{2(a^2+b^2t)^{3/2}}\,dt = \frac{1}{a}-\frac{1}{\sqrt{a^2+b^2}}\,. \tag{4.38}$$

If we use the series representation of ${}_2F_1$ in (4.38), we get

$$\begin{aligned}\int_0^1 \frac{b^2}{2a^3}\,{}_2F_1\left(\frac{3}{2};2;2;-\frac{b^2t}{a^2}\right)dt &= \int_0^1 \frac{b^2}{2a^3}\sum_{n=0}^{\infty}\frac{(3/2)_n(2)_n(-b^2t)^n}{(2)_n n!\,a^{2n}}\,dt\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n(3/2)_n b^{2(n+1)}}{2a\,n!\,a^{2(n+1)}}\int_0^1 t^n dt\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n(3/2)_n b^{2(n+1)}}{2a\,(n+1)!\,a^{2(n+1)}}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n(1/2)_{n+1} b^{2(n+1)}}{a\,(n+1)!\,a^{2(n+1)}}\end{aligned} \tag{4.39}$$

as $(a+1)_n = \frac{(a)_{n+1}}{a}$ and so $(\frac{3}{2})_n = (\frac{1}{2}+1)_n = 2(\frac{1}{2})_{n+1}$. Now, shifting the index one step backward by setting $n+1=n$, the series in (4.39) could be written as

$$-\sum_{n=1}^{\infty}\frac{(-1)^n(1/2)_n b^{2n}}{a\,n!\,a^{2n}} = \frac{1}{a}-\sum_{n=0}^{\infty}\frac{(-1)^n(1/2)_n b^{2n}}{a\,n!\,a^{2n}}\,. \tag{4.40}$$

Substitution of (4.40) in (4.39) and then in (4.38) gives

$$\int_0^1 \frac{b^2}{2a^3}\,{}_2F_1\left(\frac{3}{2};2;2;-\frac{b^2t}{a^2}\right)dt = \frac{1}{a}-\sum_{n=0}^{\infty}\frac{(-1)^n(1/2)_n b^{2n}}{a\,n!\,a^{2n}} = \frac{1}{a}-\frac{1}{\sqrt{a^2+b^2}}\,. \tag{4.41}$$

This implies that

$$\sum_{n=0}^{\infty}\frac{(-1)^n(1/2)_n b^{2n}}{a\,n!\,a^{2n}} = \frac{1}{\sqrt{a^2+b^2}}\,.$$

Or

$$\sum_{n=0}^{\infty}\frac{(1/2)_n(-b^2/a^2)^n}{n!\,a^{2n}} = \frac{a}{\sqrt{a^2+b^2}}\,. \tag{4.42}$$

The series in (4.42) is indeed both ${}_2F_1\left(\frac{1}{2};1;1;-\frac{b^2}{a^2}\right)$ and ${}_1F_0\left(\frac{1}{2};-;-;-\frac{b^2}{a^2}\right)$, and hence (4.32) is obtained as desired. □

4.4 Applications: Special Cases and Laplace Transform

In this section we will study some special cases of (4.25). For $(p=0)$, where we are back to classical Hankel's generalization (4.34), the following two cases

$$\int_0^\infty e^{-az} z^\alpha J_\alpha(bz)dz = \frac{(2b)^\alpha \Gamma(\alpha+\frac{1}{2})}{\sqrt{\pi}(a^2+b^2)^{\alpha+\frac{1}{2}}} \quad (Re(\alpha) > -\frac{1}{2}), \tag{4.43}$$

$$\int_0^\infty e^{-az} z^{\alpha+1} J_\alpha(bz)dz = \frac{2a(2b)^\alpha \Gamma(\alpha+\frac{3}{2})}{\sqrt{\pi}(a^2+b^2)^{\alpha+\frac{3}{2}}} \quad (Re(\alpha) > -1) \tag{4.44}$$

(where $\mu = \alpha+1$ and $\mu = \alpha+2$ respectively) have already been studied by Gegenbauer (1875) and observed by Sonine (1880) [37]. In our work, we study special cases where $(p \neq 0)$.

Corollary 4.4.1.

$$\int_0^\infty e^{-az} z^{-\alpha} J_{\alpha,p}(hz)dz$$

$$= \frac{h^\alpha}{a2^\alpha \Gamma(\alpha+1)} \left[1 - \frac{h^2}{2a^2(\alpha+1)} F_p\left(\frac{3}{2};1;\alpha+2;-\frac{h^2}{a^2}\right)\right]. \tag{4.45}$$

Proof. If we set $(\mu = -\alpha+1)$ in (4.25), we have

$$\int_0^\infty e^{-az} z^{-\alpha} J_{\alpha,p}(hz)dz$$

$$= \frac{h^\alpha}{a2^\alpha \Gamma(\alpha+1)} \cdot \left[1 - \frac{h^2}{2a^2(\alpha+1)} F_p\left(\begin{matrix} \frac{3}{2} & & - \\ & - & \end{matrix} ;1;\alpha+2;-\frac{h^2}{a^2}\right)\right]. \tag{4.46}$$

The GEGHF appearing on the right side of (4.46) is the EGHF and so (4.46) could simply be written as (4.45). □

One of the most important tools of applied mathematics that is widely used in engineering areas is the Laplace transform. Here we see how the extension we made for the standard BF could be applied by finding its Laplace transform where its existence is guaranteed by [27, Th 4.2] as the function (EBF) is continuous on the improper interval and by (3.67) is bounded.

It is known in the literature that the Laplace transform of $J_0(ht)$ is given by

$$\mathscr{L}\{J_0(ht);a\} = \int_0^\infty e^{-at} J_0(ht)dt = \frac{1}{\sqrt{a^2+h^2}}. \tag{4.47}$$

But how would the Laplace transform of $J_{0,p}(ht)$ look like? The following corollary answers this question.

Corollary 4.4.2.

$$\mathscr{L}\left\{J_{0,p}(ht);a\right\} = \frac{1}{a} - \frac{h^2}{2a^3} F_p\left(\frac{3}{2},1,2;-\frac{h^2}{a^2}\right). \tag{4.48}$$

Proof. By (4.46)

$$\mathscr{L}\left\{z^{-\alpha} J_{\alpha,p}(ht);a\right\} = \frac{h^\alpha}{a2^\alpha \Gamma(\alpha+1)} \cdot \left[1 - \frac{h^2}{2a^2(\alpha+2)} F_p\left(\frac{3}{2};1;\alpha+2;-\frac{h^2}{a^2}\right)\right]. \tag{4.49}$$

Hence, (4.48) is obtained by setting $\alpha = 0$ in (4.49). □

It is worth mentioning to look at (4.25) as another important transformation which is the Mellin's transform for the function $f(t) = e^{-at} J_{\alpha,p}(ht)$.

$$F(\mu) = \mathscr{M}\left(e^{-at}J_{\alpha,p}(ht)\right) = \frac{h^\alpha\Gamma(\mu+\alpha)}{a^{\alpha+\mu}2^\alpha\Gamma(\alpha+1)}$$
$$\times\left[1-\frac{h^2(\mu+\alpha+1)(\mu+\alpha)}{4a^2(\alpha+1)}F_p\left(\begin{array}{ccc}\frac{\mu+\alpha+2}{2} & & \frac{\mu+\alpha+3}{2} \\ & 2 & \end{array};1;\alpha+2;-\frac{h^2}{a^2}\right)\right]. \tag{4.50}$$

Of course, we cannot restrict any value for α in terms of μ as we have done for the Laplace transforms.

As stated in the introduction that, in potential theory, $e^{-\rho t}J_0(zt)$ is taken as potential function [37] and so we have the more general form of this which is $e^{-\rho t}J_{0,p}(zt)$ where the potential function is given by (4.22).

4.5 Asymptotic Behavior

As the asymptotic behavior of the standard BF have been studied by different methods [2, 6, 26, 28, 37, 39] and given as

$$J_p(x) \sim \sqrt{\frac{2}{\pi x}}\cos\left[z-\frac{(p+\frac{1}{2})\pi}{2}\right] \quad (x\to\infty), \tag{4.51}$$

we proceed in this section with finding the asymptotic behavior of $J_{\alpha,p}$ for large $|z|$. We find it in two ways as the following two theorems suggest.

Theorem 4.5.1.

$$J_{\alpha,p}(z) \sim \frac{z^\alpha}{2^\alpha\Gamma(\alpha+1)}\left(1-\sum_{n=0}^{\infty}\frac{(-1)^n(z/2)^{2n+2}E_{n+2}(p)}{(2)_n n!}\right) \quad (|z|\to\infty). \tag{4.52}$$

Proof. To obtain the asymptotic behavior, we start with the series representation of the EBF given by (3.46):

$$J_{\alpha,p}(z) = \frac{z^{\alpha}}{2^{\alpha}\Gamma(\alpha+1)} - \sum_{n=0}^{\infty} \frac{\beta(n+1;\alpha+1;p)(-1)^n z^{2n+\alpha+2}}{\Gamma(\alpha+1)2^{2n+\alpha+2}(2)_n n!}.$$

So, basically, we will work with the series part of the upper representation where we write the extended beta function in its integral representation given by (1.15) to have

$$\sum_{n=0}^{\infty} \frac{\beta(n+1;\alpha+1;p)(-1)^n z^{2n}}{2^{2n}(2)_n n!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{2^{2n}(2)_n n!} \int_0^1 t^n (1-t)^{\alpha} \, exp\left[\frac{-p}{t(1-t)}\right] dt. \quad (4.53)$$

Set $\left(t = \frac{\tau}{z}\right)$ in the integral on the right side of (4.53) to get it as

$$\int_0^z \tau^n \left(1 - \frac{\tau}{z}\right)^{\alpha} exp\left[\frac{-pz}{\tau(1-\frac{\tau}{z})}\right] d\tau. \quad (4.54)$$

Then, as $(|z| \to \infty)$ we have $\left(\frac{\tau}{z} \ll 1\right)$ and so,

$$\sum_{n=0}^{\infty} \frac{\beta(n+1;\alpha+1;p)(-1)^n z^{2n}}{2^{2n}(2)_n n!} \sim \sum_{n=0}^{\infty} \frac{(-1)^n z^{n-1}}{2^{2n}(2)_n n!} \int_0^z \tau^n \, exp\left(\frac{-pz}{\tau}\right) d\tau. \quad (4.55)$$

The integral in the right side of (4.55) can be evaluated by setting $c = pz$ in (A.32) so as to have

$$\int_0^z \tau^n \, exp\left(\frac{-pz}{\tau}\right) d\tau = z^{n+1} E_{n+2}(p). \quad (4.56)$$

Hence, (4.52) is obtained by substituting (4.56) in (4.55) and then in (3.46). □

Corollary 4.5.2.

$$J_{0,p}(z) \sim 1 - \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n+2} E_{n+2}(p)}{(2)_n n!} \qquad (|z| \to \infty). \quad (4.57)$$

Proof. Set $\alpha = 0$ in (4.52) to obtain (4.57). □

The other asymptotic behavior we might see on the EBF is found by the use of the integral representation (3.59).

Theorem 4.5.3.

$$J_{\alpha,p}(z) \sim \frac{\left(\frac{z}{2}\right)^{\alpha}}{\Gamma(\alpha+1)} \quad (|z| \to \infty). \tag{4.58}$$

Proof. Unlike the standard BF [2, 28, 37], the EBF will behave as in (4.58). This is suggested by the integral representation (3.59)

$$J_{\alpha,p}(z) = \frac{(z/2)^{\alpha}}{\Gamma(\alpha+1)} \left[1 - \frac{z}{2}\int_0^1 \frac{(1-t)^{\alpha}}{\sqrt{t}} \, exp\left[\frac{-p}{t(1-t)}\right] J_1(z\sqrt{t})dt\right]$$

for sufficiently large values of z and the speed of this behavior depends on the values of p chosen. To see this, we note that our function consists of two terms that differ by a negative sign. What would shrink with the high values of p is the integral part and hence we are remained with the first term which is $\dfrac{\left(\frac{z}{2}\right)^{\alpha}}{\Gamma(\alpha+1)}$. □

If we look closely to the integral in the second term of (4.17), we find that there are four factors that govern its behavior. These factors are $(1-t)^{\alpha}, \sqrt{t}, J_1(z\sqrt{t})$ and the exponent term. For $(-1 < \alpha < 0)$, we have two poles at the end points of the integral. For $(\alpha \geq 0)$, we have only one pole at root t. The problem of the growth of the integrand will occur only at the end points of the integral where we have poles. For $(p = 0)$, we are back to the standard BF where the effect of the integral will not be negligible at any stage of its run.

However, for non zero p, the exponent will absorb such growth of the integrand at the end points since the exponent is going to be exponent of $(-\infty)$ and the situation of the whole function will differ here. This exponent factor is actually the regularizer factor added to the standard beta function $\beta(x,y)$ which is not defined at $(x = y = 0)$. *This exponent factor will play the same role here. It will absorb the integrand strongly.*

For small values of p, the exponential factor will be closer to one and the situation will be closer to standard BF but only to a certain stage since $J_1(z)$ will go to zero for sufficiently large z. So, we have the exponential term that will absorb the blowup of the integral at the end points and we have $J_1(z)$ that will absorb the integral for large values of z. Hence, the integral is going to be very close to zero at a certain stage and so we have (4.58) at that stage.

Therefore, the speed of the shrinkage of the integral term depends on values of p chosen. The higher the values of p, the faster the asymptotic behavior appears on the function. No matter how small p is, the function as a whole will go to the asymptotic term at a certain stage. Indeed for sufficiently small values of p, the integral will not die down so fast and hence the whole function will be as the standard BF.

This might not be seen clearly by numerical computations since for sufficiently small values of p, the precisions of the assigned values will be considered as zero by computers and hence the exponent as one. This is numerical problem but not analysis problem. However, as long as the precisions are considered, the asymptotic behavior will appear.

On the other hand, for the values of p which are not so small, the deviation from the standard BF is clearer at early stages.

The following graphs explain the matter where in all graphs we have taken the following values for $\alpha : (\alpha = 0, 0.25, 0.5, 0.75, 1)$

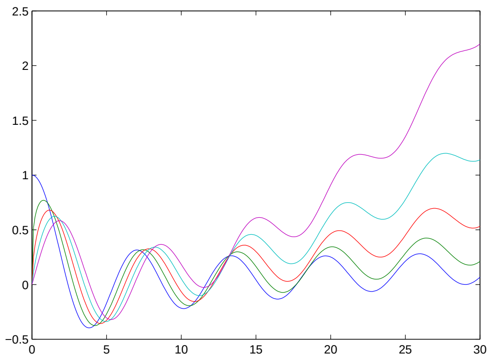

Figure 4.1: Plot of the EBF for ($p = 0.2E-3$).

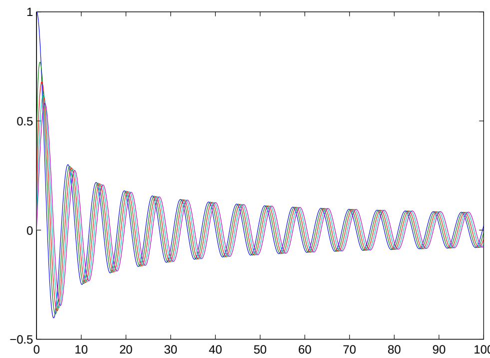

Figure 4.2: Plot of the EBF for ($p = 0.2E-8$).

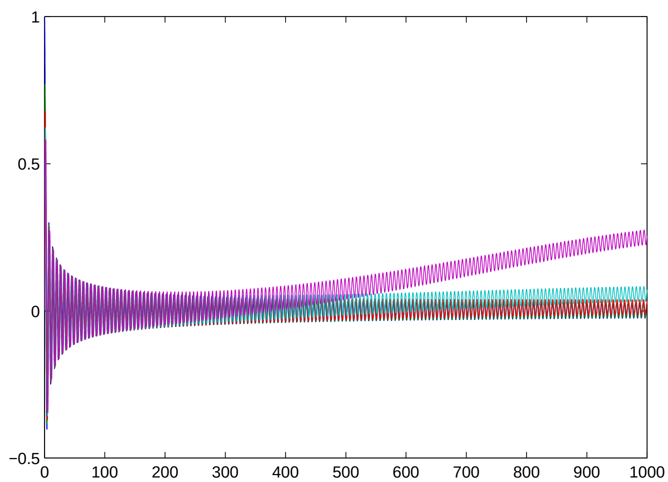

Figure 4.3: Plot of the EBF for ($p = 0.2E-3$).

It is clear from Figure 4.1 that the deviation from standard BF's behavior started at ($z \approx 13$) which is in a very early stage if compared with that where ($p = 0.2E-8$). When ($p = 0.2E-8$), the deviation was not clear when we toke the interval on $(0, 100)$ as in Figure 4.2. However, when the interval was enlarged as in Figure 4.3, the deviation appeared to be clear at ($z \approx 250$) and the behavior started to go asymptotic as (4.58) stated.

4.6 Mellin-Barnes Contour Integral for the EBF

In this section we develop the Mellin-Barnes contour integral for the EBF in the form of the following theorem.

Theorem 4.6.1. *The Mellin-Barnes contour integral for $J_{\alpha,p}(z)$ is give as*

$$J_{\alpha,p}(z) = \frac{z^{\alpha}}{2^{\alpha}\Gamma(\alpha+1)}\left[1 - \frac{1}{2\pi i}\oint_{c}\frac{\Gamma(-s)\beta(s+1,\alpha+1;p)(z/2)^{2s+2}}{2^{2s}\Gamma(2+s)}ds\right], \tag{4.59}$$

where C is a loob passing from $(+\infty)$ in the positive sense, encircling the pools of $\Gamma(-s)$ *and going back to $(+\infty)$.*

Proof. As the series representation of $J_{\alpha,p}(s)$ is given by (3.46) as

$$J_{\alpha,p}(z) = \frac{z^{\alpha}}{2^{\alpha}\Gamma(\alpha+1)} - \frac{z^{\alpha+2}}{2^{\alpha+2}\Gamma(\alpha+1)}\sum_{n=0}^{\infty}\frac{\beta(n+1,\alpha+1;p)(-1)^{n}z^{2n}}{\Gamma(\alpha+1)2^{2n}\Gamma(n+2)n!}$$

$$(Re(\alpha) > -1),$$

consider the following contour integral

$$I = \frac{1}{2\pi i}\oint_{c}\frac{\Gamma(-s)\beta(s+1,\alpha+1;p)z^{2z}}{2^{2s}\Gamma(2+s)}ds, \tag{4.60}$$

where C is a loob passing from $(+\infty)$ in the positive sense, encircling the pools of $\Gamma(-s)$ and going back to $(+\infty)$. If we use Rule 2 (see the appendix), we find that the number of the gamma function with negative multiplicity in the numerator is 1 and that with positive multiplicity in the denominator is again 1 and so they both sum to 2 while the number of the gamma function with positive multiplicity in the numerator and with negative multiplicity in the denominator is zero and so they both sum to 0. This gives the converges of I to be valid for all finite non zero z. i.e. $(|z| < \infty, z \neq 0)$.

Now, if we consider I by noting that the extended beta function has no finite singularities inside C, then I can be evaluated by Cauchy's Residue Theorem (A.34) where by (A.35) we find the sum of the residues of $\Gamma(-s)$ are given as

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$$

and hence

$$I = \sum_{n=0}^{\infty} \frac{(-1)^n \beta(n+1, \alpha+1; p) z^{2n}}{2^{2n}\Gamma(n+2) n!} = \sum_{n=0}^{\infty} \frac{(-1)^n \beta(n+1, \alpha+1; p) z^{2n}}{2^{2n}(2)_n n!}. \tag{4.61}$$

This gives the Mellin Contour integral of $J_{\alpha,p}(z)$ to be as in (4.59) as desired. □

To proceed further with finding the asymptotic behavior of $J_{\alpha,p}$ for large $|z|$, we find that the integral in I has no poles to the left of poles of $\Gamma(-s)$ and so shifting the loop to the left yields no useful asymptotic information. So, we use the reflection formula [28, Eq 2.1.20] for $\Gamma(-s)$ could be written as

$$\Gamma(-s) = \frac{\pi}{\Gamma(s+1)\sin\pi(-s)} = \frac{-\pi}{\Gamma(s+1)\sin\pi s}.$$

This gives

$$I = \frac{-\pi}{2\pi i} \oint_c \frac{\beta(s+1, \alpha+1; p) z^{2s}}{2^{2s}\Gamma(s+1)\Gamma(s+2)\sin\pi s} ds.$$

Or,

$$I = \frac{-1}{2i} \oint_c \frac{\beta(s+1, \alpha+1; p) z^{2s}}{2^{2s}\Gamma(s+1)\Gamma(s+2)\sin\pi s} ds. \tag{4.62}$$

So, we stop here since we do not have powerful results to evaluate (4.62) and to have abound for the remainder term that would result in the process of evaluation. We leave this part for further research work that might be done in the future.

4.7 Concluding Remarks and Observations

In this chapter, we have found the Lipschitz integral for the EBF. Then, we generalized the result to Hankel's generalization and we found that the Hankel's generalization is expressible in terms of the GEGHF that we introduced in this work.

As the area of Laplace transforms is one of the most important areas in different fields of science, we found the Laplace transform for the EBF of order zero in a closed mathematical form. This is written in terms of a recently extended function which is the EGHF that again appeared as a special case of the GEGHF.

Asymptotic behavior of the EBF has also been studied in this chapter. We provided two asymptotic behaviors. One was in terms of the exponential integral function. The other was by using the integral representation of the EBF. Then we finished the work by finding the Mellin-Barnes integral.

Chapter 5

On the Differential Equation of the Extended Bessel Function of the First Kind

As the importance of special functions arises where their applications arise, the importance of developing a differential equation becomes apparent. It is actually the crucial point in the application fields. Since in real life problems, a given problem most often ends up with a differential equation, the question what function would solve the problem rises up. Of course, different ways are developed to solve such differential equations. However, *finding a specific closed form solution in terms of a function is always the best* since in many cases series solutions are found where no closed forms are apparently possible. In such cases, estimations are applied.

So, the question that always comes on air has two directions. One direction is that where we have the differential equation and we seek its solutions in terms of a specific special function. The other is that if we have a new special function, what is the differential equation that such a special function is a solution of? The answer to both questions remains open for researchers to work on depending on what problem they are dealing with. However, it is good to remark on the second direction by noting that finding a differential equation for a newly defined special function means a trial on finding where such a function would possibly applied.

From this introduction we will see that in this chapter we are working on the second direction since our function under study which is the EBF is newly introduced to the literature. Although the differential equation we developed:

$$z^2y''(z)-(2\alpha+1)zy'(z)+(\alpha^2+2\alpha+z^2)y(z)=2(\alpha+1)zJ_{\alpha+1,p}(z) \tag{5.1}$$

and going to explain in section three is a non-homogenous differential equation for the EBF, we call it *a generalization* of the well known Bessel differential equation:

$$z^2y''(z)+zy'(z)+(z^2-\alpha^2)y(z)=0 \tag{5.2}$$

that the standard BF satisfies. We call it so because of two reasons. *First*, if we set the parameter ($p=0$) in (5.1), then it is no more non-homogeneous and it is *exactly (5.2)* as we will see where $y(z)=J_\alpha(z)$. The second reason is that $J_{\alpha,p}(z)$ *(the EBF of the first kind) is a particular solution of (5.1).*

We should comment here that if a homogenous differential equation for a specific special function does not seem to be feasible in the closed horizon, this does not mean that such a function is not applicable in real applied problems and so such a function is not so worthy!. It could be, however, a particular solution of a non-homogeneous problem where it might be impossible to find a particular solution in terms of the simple well known special functions as in (5.1). Moreover, it could simplify a complicated form and give it in a nice elegant form which is most often the case [11, 16].

In this chapter we start section two by introducing a function $f(m,\alpha,z;p)$ that would help in finding some new recurrence relation for the EBF and then will be used to solve (5.1). In section three, we find a particular solution for (5.1) which is the main target of this chapter. We finish the chapter with a concluding section.

5.1 Recurrence Relations for the EBF

The series representation of $J_{\alpha,p}(z)$ given in (3.46) could be written as:

$$J_{\alpha,p}(z) = \frac{z^{\alpha}}{2^{\alpha}\Gamma(\alpha+1)} - \frac{z^{\alpha+2}}{2^{\alpha+2}\Gamma(\alpha+1)} \sum_{n=0}^{\infty} \frac{\beta(n+1;\alpha+1;p)(-1)^n z^{2n}}{2^{2n}\Gamma(n+2)n!}$$

$$(Re(\alpha) > -1), \tag{5.3}$$

where

$$\beta(x,y;p) = \int_0^1 t^{x-1}(1-t)^{y-1}\, exp\left[\frac{-p}{t(1-t)}\right] dt \quad (Re(p) > 0).$$

is the extended beta function introduced in (1.15).

We define the auxiliary function $f(m,\alpha,z;p)$ as

$$f(m,\alpha,z;p) = \sum_{n=0}^{\infty} \frac{(-1)^n \beta(n+m,\alpha;p)(z/2)^{2n}}{(2)_n n!}. \tag{5.4}$$

This function is well defined and convergent for all z and uniformly convergent for $|z| < 2$, and also is symmetric in z. This function is directly related to the EBF and is going to be used to develop some recurrence relations on the EBF as well as finding the differential equation that the EBF satisfies.

From (5.3),

$$J_{\alpha+m,p}(z) = \frac{(z/2)^{\alpha+m}}{\Gamma(\alpha+m+1)}$$
$$\times\left(1-\sum_{n=0}^{\infty}\frac{(-1)^n\beta(n+1,\alpha+m+1;p)(z/2)^{2n+2}}{(2)_n n!}\right)$$
$$= \frac{(z/2)^{\alpha+m}}{(\alpha+1)_m\Gamma(\alpha+1)}$$
$$\times\left(1-\sum_{n=0}^{\infty}\frac{(-1)^n\beta(n+1,\alpha+m+1;p)(z/2)^{2n+2}}{(2)_n n!}\right) \quad (m\geq 0). \tag{5.5}$$

Just note that

$$J_{\alpha,p}(z) = \frac{(z/2)^{\alpha}}{\Gamma(\alpha+1)}\left[1-\left(\frac{z}{2}\right)^2 f(1,\alpha+1,z;p)\right] \tag{5.6}$$

and

$$J_{\alpha+m,p}(z) = \frac{(z/2)^{\alpha+m}}{(\alpha+1)_m\Gamma(\alpha+1)}\left[1-\left(\frac{z}{2}\right)^2 f(1,\alpha+m+1,z;p)\right]. \tag{5.7}$$

From (5.3) and (5.5),

$$J_{\alpha+m,p}(z) - \frac{(z/2)^m}{(\alpha+1)_m}J_{\alpha,p}(z) = \frac{(z/2)^{\alpha+m+2}}{(\alpha+1)_m\Gamma(\alpha+1)}$$

$$\times\sum_{n=0}^{\infty}\frac{(-1)^n(z/2)^{2n}}{(2)_n n!}\left[\beta(n+1,\alpha+1;p)-\beta(n+1,\alpha+m+1;p)\right]. \tag{5.8}$$

We will use two ways to evaluate the difference between the two extended beta functions in (5.8).

First, we will use the integral representation for the extended beta function. This can be:

$$\beta(n+1,\alpha+1;p)-\beta(n+1,\alpha+m+1;p)$$

$$= \int_0^1 t^n(1-t)^{\alpha} exp\left[\frac{-p}{t(1-t)}\right]\left[1-(1-t)^m\right]dt. \tag{5.9}$$

Closed forms of this integral could be found for special cases as $(m = 1\,or\,2)$ but higher ordered are not so elegant. So, for $(m = 1)$,

$$\beta(n+1,\alpha+1;p)-\beta(n+1,\alpha+2;p)=\beta(n+2,\alpha+1;p) \tag{5.10}$$

The second way will be the use of the result (2.24). This gives:

$$\beta(n+1,\alpha+1;p)-\beta(n+1,\alpha+m+1;p)=\beta(n+2,\alpha;p)-\beta(n+m+2,\alpha;p). \tag{5.11}$$

which if put in (5.8) implies the following general formula:

$$J_{\alpha+m,p}(z)-\frac{(z/2)^m}{(\alpha+1)_m}J_{\alpha,p}(z)=\frac{(z/2)^{\alpha+m+2}}{(\alpha+1)_m\Gamma(\alpha+1)}[f(2,\alpha,z;p)-f(2+m,\alpha,z;p)]. \tag{5.12}$$

For $m = 1$, (5.12) gives

$$J_{\alpha+1,p}(z)-\frac{(z/2)}{\alpha+1}J_{\alpha,p}(z)=\frac{(z/2)^{\alpha+3}}{\Gamma(\alpha+2)}[f(2,\alpha,z;p)-f(3,\alpha,z;p)]. \tag{5.13}$$

However, (5.13) can be written in a different way if we use (5.10) in (5.8) to give

$$J_{\alpha+1,p}(z)-\frac{(z/2)}{\alpha+1}J_{\alpha,p}(z)=\frac{(z/2)^{\alpha+3}}{\Gamma(\alpha+2)}f(2,\alpha+1,z;p). \tag{5.14}$$

And so,

$$f(2,\alpha+1,z;p)=f(2,\alpha,z;p)-f(3,\alpha,z;p). \tag{5.15}$$

Set $(\alpha \to \alpha-1)$ in (5.14) and multiply the resultant equation with the factor $\frac{(z/2)}{\alpha+1}$ and then add it to (5.14) again to get

$$J_{\alpha+1,p}(z)-\frac{(z/2)^2}{\alpha(\alpha+1)}J_{\alpha-1,,p}(z)=\frac{(z/2)^3}{\Gamma(\alpha+2)}[f(2,\alpha,z;p)+f(2,\alpha+1,z;p)]. \tag{5.16}$$

By using (5.15) in (5.16) we obtain

$$J_{\alpha+1,p}(z)-\frac{(z/2)^2}{\alpha(\alpha+1)}J_{\alpha-1,p}(z)=\frac{(z/2)^3}{\Gamma(\alpha+2)}[2f(2,\alpha,z;p)-f(3,\alpha,z;p)]. \tag{5.17}$$

On the other hand, if we $(\alpha \to \alpha-1)$ in (5.14) and multiply the resultant equation with the factor $\frac{(z/2)}{\alpha+1}$ and then subtract it from (5.14) again, we get:

$$J_{\alpha+1,p}(z)-\frac{z}{\alpha+1}J_{\alpha,p}(z)+\frac{(z/2)^2}{\alpha(\alpha+1)}J_{\alpha-1,p}(z)=\frac{(z/2)^3}{\Gamma(\alpha+2)}[f(2,\alpha+1,z;p)+f(2,\alpha,z;p)]. \tag{5.18}$$

By applying (5.15) we obtain

$$J_{\alpha+1,p}(z)-\frac{z}{\alpha+1}J_{\alpha,p}(z)+\frac{(z/2)^2}{\alpha(\alpha+1)}J_{\alpha-1,p}(z)=-\frac{(z/2)^3}{\Gamma(\alpha+2)}f(3,\alpha,z;p). \tag{5.19}$$

Note also that by using [11, eq. (5.62)]

$$\frac{\partial}{\partial p}J_{\alpha,p}(z)=\frac{(z/2)^{\alpha+2}}{\Gamma(\alpha+1)}\sum_{n=0}^{\infty}\frac{(-1)^n(z/2)^{2n}\beta(n,\alpha;p)}{(2)_n n!} \tag{5.20}$$

This gives $\frac{\partial}{\partial p}J_{\alpha,p}(z)$ in terms of the newly defined function $f(m,\alpha,z;p)$ as

$$\frac{\partial}{\partial p}J_{\alpha,p}=\frac{(z/2)^{\alpha+2}}{\Gamma(\alpha+1)}f(0,\alpha,z;p). \tag{5.21}$$

5.2 Solutions of the Non-Homogeneous Differential Equation (5.1)

In the previous section we defined the function $f(m,\alpha,z;p)$ and we utilized it to develop some new recurrence relations on the EBF. Here, in this section we use this function to prove that the EBF is a particular solution of the non-homogeneous part of (5.1) and then we find its general solution.

Theorem 5.2.1. *$J_{\alpha,p}(z)$ is a particular solution for the non-homogeneous equation (5.1).*

Proof. From (5.6) we have,

$$f(1,\alpha+1,z;p) = \left(\frac{2}{z}\right)^2 - \left(\frac{2}{z}\right)^{\alpha+2} \Gamma(\alpha+1) J_{\alpha,p}(z) \tag{5.22}$$

and

$$f(1,\alpha,z;p) = \left(\frac{2}{z}\right)^2 - \left(\frac{2}{z}\right)^{\alpha+1} \Gamma(\alpha) J_{\alpha-1,p}(z)\,. \tag{5.23}$$

Define the operator $\theta = z\frac{\partial}{\partial z}$. Then,

$$\begin{aligned}
\theta f(1,\alpha+1,z;p) &= \sum_{n=0}^{\infty} \frac{2n(-1)^n z^{2n} \beta(n+1,\alpha+1;p)}{2^{2n}(2)_n n!} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n z^{2n} \beta(n+1,\alpha+1;p)}{2^{2n-1}(n-1)!(2)_n}\,.
\end{aligned} \tag{5.24}$$

As $(2)_{n+1} = (2+n)(2)_n$, then shifting the index $(n \to n+1)$ in (5.24) gives

$$\theta f(1,\alpha+1,z;p) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} z^{2n+2} \beta(n+2,\alpha+1;p)}{2^{2n+1} n!(2+n)(2)_n}\,. \tag{5.25}$$

Also we have

$$\begin{aligned}
\theta^2 f(1,\alpha+1,z;p) &= \sum_{n=1}^{\infty} \frac{2n(-1)^n z^{2n} \beta(n+1,\alpha+1;p)}{2^{2n-1}(n-1)!(2)_n} \\
&= \sum_{n=0}^{\infty} \frac{(2n+2)(-1)^{n+1} z^{2n+2} \beta(n+2,\alpha+1;p)}{2^{n+1} n!(2+n)(2)_n}\,.
\end{aligned} \tag{5.26}$$

So,

$$\theta(\theta+2)f(1,\alpha+1,z;p) = \sum_{n=0}^{\infty} \frac{2(n+2)(-1)^{n+1}z^{2n+2}\beta(n+2,\alpha+1;p)}{2^{n+1}n!(2+n)(2)_n}$$
$$= -z^2 \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}\beta(n+2,\alpha+1;p)}{2^{2n}n!(2)_n}$$
$$= -z^2 f(2,\alpha+1,z;p). \tag{5.27}$$

Now, in order to write (5.27) in terms of the EBF, we apply (5.22) on the left side and (5.14) on the right side. By applying (5.14), the right side becomes:

$$\begin{aligned} \text{right side} &= -z^2 f(2,\alpha+1,z;p) \\ &= -z^2 \left[\left(\frac{2}{z}\right)^{\alpha+3} \Gamma(\alpha+2) J_{\alpha+1,p}(z) - \left(\frac{2}{z}\right)^{\alpha+2} \Gamma(\alpha+1) J_{\alpha,p}(z) \right] \\ &= -2^{\alpha+2} z^{-\alpha} \Gamma(\alpha+1) J_{\alpha,p}(z) + \frac{2^{\alpha+3}}{z^{\alpha+1}} \Gamma(\alpha+2) J_{\alpha+1,p}(z) \\ &= -[2^{\alpha+3} z^{-\alpha-1} \Gamma(\alpha+2) J_{\alpha+1,p}(z) - 2^{\alpha+2} z^{-\alpha} \Gamma(\alpha+1) J_{\alpha,p}(z)] \\ &= -2^{\alpha+2} \Gamma(\alpha+1) \left[2z^{-\alpha-1}(\alpha+1) J_{\alpha+1,p}(z) - z^{-\alpha} J_{\alpha,p}(z) \right]. \end{aligned} \tag{5.28}$$

By applying (5.22) on the left side we have

$$\begin{aligned} \text{left side} &= \theta(\theta+2) f(1,\alpha+1,z;p) \\ &= \theta(\theta+2) \left[\left(\frac{2}{z}\right)^2 - \left(\frac{2}{z}\right)^{\alpha+2} \Gamma(\alpha+1) J_{\alpha,p}(z) \right]. \end{aligned} \tag{5.29}$$

First, we note that $\theta(\theta+2)$ annihilates $\left(\frac{2}{z}\right)^2$; that is

$$\theta(\theta+2)\left(\frac{2}{z}\right)^2 = \theta(\theta+2)\left(\frac{4}{z^2}\right) = 0. \tag{5.30}$$

This is because

$$\theta\left(\frac{4}{z^2}\right) = z\frac{\partial}{\partial z}(4z^{-2}) = -8z^{-2}.$$
$$2\theta\left(\frac{4}{z^2}\right) = -16z^{-2}.$$
$$\theta^2\left(\frac{4}{z^2}\right) = \theta(-8z^{-2}) = 16z^{-2}.$$

So, in (5.29) we proceed with $\theta(\theta+2)\left[z^{-\alpha-2}J_{\alpha,p}(z)\right]$ multiplied with the factor $\left(-2^{\alpha+2}\Gamma(\alpha+1)\right)$.

$$\theta\left(z^{-\alpha-2}J_{\alpha,p}(z)\right) = -(\alpha+2)z^{-\alpha-2}J_{\alpha,p}(z) + z^{-\alpha-1}J'_{\alpha,p}(z). \tag{5.31}$$

$$\begin{aligned}\theta^2\left(z^{-\alpha-2}J_{\alpha,p}(z)\right) =& (\alpha+2)^2z^{-\alpha-2}J_{\alpha,p}(z) - (2\alpha+3)z^{-\alpha-1}J'_{\alpha,p}(z)\\ &+ z^{-\alpha}J''_{\alpha,p}(z).\end{aligned} \tag{5.32}$$

Therefore,

$$\begin{aligned}(\theta^2+2\theta)\left[z^{-\alpha-2}J_{\alpha,p}(z)\right] =& (\alpha^2+2\alpha)z^{-\alpha-2}J_{\alpha,p}(z) - (2\alpha+1)z^{-\alpha-1}J'_{\alpha,p}(z)\\ &+ z^{-\alpha}J''_{\alpha,p}(z)\end{aligned} \tag{5.33}$$

and hence, by (5.33), (5.29) becomes

$$\begin{aligned}\text{left side} =& -2^{\alpha+2}\Gamma(\alpha+1)\\ &\times\left[(\alpha^2+2\alpha)z^{-\alpha-2}J_{\alpha,p}(z) - (2\alpha+1)z^{-\alpha-1}J'_{\alpha,p}(z) + z^{-\alpha}J''_{\alpha,p}(z)\right].\end{aligned} \tag{5.34}$$

Equating (5.28) and (5.34) we obtain:

$$z^2J''_{\alpha,p}(z) - (2\alpha+1)zJ'_{\alpha,p}(z) + (\alpha^2+2\alpha+z^2)J_{\alpha,p}(z) = 2(\alpha+1)zJ_{\alpha+1,p}(z). \tag{5.35}$$

as desired. □

In the sense we explained in the introduction, (5.35) is indeed a generalization of the differential equation of the standard BF which is given by [2, 37]

$$z^2 J''_\alpha(z) + zJ'_\alpha(z) + (z^2 - \alpha^2)J_\alpha(z) = 0 \tag{5.36}$$

when putting $p = 0$. To see that, simply use the identity [2, eq (6.15)]

$$zJ_{\alpha+1}(z) = \alpha J_\alpha(z) - zJ'_\alpha(z) \tag{5.37}$$

on the right side of (5.35) with $p = 0$ to get

$$z^2 J''_\alpha(z) - (2\alpha+1)zJ'_\alpha(z) + (\alpha^2 + 2\alpha + z^2)J_\alpha(z) = 2(\alpha+1)[\alpha J_\alpha(z) - zJ'_\alpha(z)]. \tag{5.38}$$

Rearranging terms of (5.38) gives (5.36).

5.3 Solutions of the Homogeneous Differential Equation (5.1)

As the homogenous part of (5.1) is a classical second order differential equation and its solution is guaranteed since ($x = 0$) is a regular singular point [40, TH6.2], the following theorem provide its solution as a power series solution by using the well known Frobenius method [27, 40].

Theorem 5.3.1. *The general solution of the homogeneous part of (5.1) is given by*

$$y(z) = z^{\alpha+1}[AJ_\nu(z) + BY_\nu(z)] \tag{5.39}$$

where

$$Y_\nu(z) = \frac{\cos \nu\pi \, J_\nu(z) - J_{-\nu}(z)}{\sin \nu\pi} \tag{5.40}$$

is Neumann's function or the Bessel function of the second kind of order ν.

Proof. By following the Frobenius method to solve the homogeneous part of (5.1), we set

$$y(z) = \sum_{n=0}^{\infty} c_n z^{n+r}. \tag{5.41}$$

This gives

$$y'(z) = \sum_{n=0}^{\infty} c_n (n+r) z^{n+r-1}$$

and so

$$y''(z) = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) z^{n+r}.$$

Substituting with the above in (5.1) gives:

$$\begin{gathered} c_0 r(r+1)z^r - (2\alpha+1)c_0 r z^r + (\alpha^2+2\alpha)c_0 z^r + \\ c_1 r(r+1)z^{r+1} - (2\alpha+1)c_1(r+1)z^{r+1} + (\alpha^2+2\alpha)c_1 z^{r+1} + \\ \sum_{n=0}^{\infty} [(n+r+2)(n+r+1) - (2\alpha+1)(n+r+2) + (\alpha^2+2\alpha]c_{n+2} + c_n] z^{n+r+2}. \end{gathered} \tag{5.42}$$

For the lowest powers of z in (5.42), we have the following equation

$$c_0 z^r [r(r+-1) - (2\alpha+1)r + \alpha^2 + 2\alpha] = 0.$$

Since nothing is gained if we set $c_0 = 0$, we will have the following indicial equation

$$r^2 - (2\alpha + 2)r + \alpha^2 + 2\alpha = 0.$$

This letter equation has two solutions, namely: $r_1 = \alpha + 2$ and $r_2 = \alpha$. Although these two roots differ by the integer 2 where two series solutions are not guaranteed [40], we will find a general series solution in a closed mathematical form.

If we use the root $r_2 = \alpha$ in the terms with $c_1 z^{r+1}$ in (5.42), it will give $-c_1 z^{\alpha+1} = 0$ and so $c_1 = 0$. Also, if we use it in the terms with z^{n+r+2}, it will give:

$$c_{n+2}[n(n+2)] = c_n. \tag{5.43}$$

This implies that $c_0 = 0$. Also, since $c_1 = 0$, then by (5.43) we have $c_3 = c_5 = 0$ and so all the odd orders. Now for even orders, set $n + 2 = 2k$ in (5.43) to get

$$c_{2k} = -\frac{c_{2k-2}}{2^2 k(k-1)} \quad (k = 2, 3, 4, ...,) \tag{5.44}$$

where c_2 is arbitrary chosen. To determine the even coefficients, we go as follows:

$$c_4 = -\frac{c_2}{2^2 \cdot 2}.$$

$$c_6 = \frac{c_2}{2^4 \cdot 3 \cdot 2 \cdot 2}.$$

$$c_8 = -\frac{c_2}{2^6 \cdot 4 \cdot 3 \cdot 2 \cdot 3 \cdot 2}.$$

$$c_{2k} = \frac{(-1)^{k-1} c_2}{2^{2k-2} k!(k-1)!} \quad (k = 2, 3, 4, ...,). \tag{5.45}$$

If we put (5.45) in (5.41) we get:

$$y(z) = \sum_{k=2}^{\infty} c_{2k} z^{2k+\alpha} = z^{\alpha} \sum_{k=2}^{\infty} \frac{(-1)^{k-1} z^{2k} c_2}{2^{2k-2} k!(k-1)!}. \tag{5.46}$$

If we shift the index in (5.46) two steps backward by setting $k = n+2$, then it is written as

$$y(z) = z^{\alpha} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} z^{2n+4} c_2}{2^{2n+2} (n+2)!(n+1)!}. \tag{5.47}$$

Now, even if we set $c_2 = 1$ and $(n+2)! = \Gamma(n+3)$ so that we have

$$y(z) = z^{\alpha+2} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} z^{2n+2}}{2^{2n+2} (n+1)! \Gamma(n+3)}, \tag{5.48}$$

(5.48) is not reducible to a known mathematical closed form. On the other hand, using the root $r_1 = \alpha + 2$ will result in an elegant closed form solution as we will see in the following discussion.

Setting $r_1 = \alpha + 2$ in the terms with $c_1 z^{r+1}$ in (5.42) will give $c_1 = 0$. Also, if we use it in the terms with z^{n+r+2}, it will give:

$$c_{n+2} = -\frac{c_n}{(n+2)(n+4)}. \tag{5.49}$$

Since $c_1 = 0$, we have by (5.49) that $c_3 = c_5 = 0$ and so all the odd orders. Again, for even orders, set $n+2 = 2k$ in (5.49) to get

$$c_{2k} = -\frac{c_{2k-2}}{4k(k+1)} \quad (k = 1, 2, 3, ...,). \tag{5.50}$$

To determine the even coefficients, we go as follows:

$$c_2 = -\frac{c_0}{2^2 \cdot 2}.$$

$$c_4 = \frac{c_0}{2^2 \cdot 2 \cdot 3}.$$

$$c_{2k} = \frac{(-1)^k c_0}{2^{2k} k!(2 \cdot 3 \cdot 4 \cdots)(k+1)} \quad (k = 1, 2, 3, ...,). \tag{5.51}$$

If we choose $c_0 = \frac{1}{2}$, then (5.51) could be written as

$$c_{2k} = \frac{(-1)^k}{2^{2k+1} k! \Gamma(k+2)} \quad (k = 1, 2, 3, ...,). \tag{5.52}$$

Now putting (5.52) in (5.41) we get:

$$y(z) = \sum_{n=0}^{\infty} c_{2n} z^{2n+\alpha+2} = z^{\alpha+1} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{2^{2n+1} n! \Gamma(n+2)}, \tag{5.53}$$

which of course gives:

$$y(z) = z^{\alpha+1} J_1(z). \tag{5.54}$$

(5.54) is indeed a solution of the homogeneous part of (5.1) and if we substitute it there, we find the following surprising result:

$$z^2[z^{\alpha+1}J_1(z)]'' - (2\alpha+1)z[z^{\alpha+1}J_1(z)]' + (\alpha^2 + 2\alpha + z^2)[z^{\alpha+1}J_1(z)] =$$

$$z^{\alpha+1}[z^2 J_1''(z) + zJ_1'(z) + (z^2 - 1)J_1(z)] = 0. \tag{5.55}$$

(5.55) actually shows us that not only $y(z) = z^{\alpha+1}J_1(z)$ is a solution of the homogeneous part of (5.1) but

$$y(z) = z^{\alpha+1} J_\nu(z) \tag{5.56}$$

(in general) is also a general solution of it. Moreover, if we use the classical discussion of the standard Bessel equation, we find that

$$y(z) = z^{\alpha+1}[AJ_\nu(z) + BJ_{-\nu}(z)] \tag{5.57}$$

is a general solution of the homogeneous part of (5.1) for 2ν is non-integer as J_ν and $J_{-\nu}$ are linearly independent in such a case. And to have the general solution for any values of ν, we have (5.39) as desired. □

Corollary 5.3.2. *The general solution of (5.1) is given as*

$$y(z) = z^{\alpha+1}[AJ_\nu(z) + BY_\nu(z)] + J_{\alpha,p}(z). \tag{5.58}$$

Proof. Combination of Theorem 5.2.1 and Theorem 5.3.1 gives (5.58). □

Remark 1: Equations (5.39), (5.56) and (5.57) shows us that the homogeneous part of (5.1) is actually a transformation of the standard Bessel equation. This transformed part together with the nonhomogeneous part form what we called the generalized Bessel differential equation whose particular solution is the extended Bessel function of the first kind.

Remark 2: Although in solving the homogeneous part of (5.1) we have $r_2 = \alpha$ which gave a series solution but not in a closed mathematical form, we were able to get a general solution in a closed mathematical form given in (5.39) by using the other root $r_1 = \alpha + 2$.

5.4 Concluding Remarks and Observations

In this chapter, we have found some new recurrence relations that seem not to be recognized in the literature. We found that they are related through our introduced function $f(m,\alpha,z;p)$. As we noticed that the left side of (5.1) does not incorporate the parameter p, the guaranteed power series solution does not include in it and so the solution of the homogenous part does not recommend any of the extended functions that are newly introduced in the literature [10, 11]. Then we used the operator $\theta = z\frac{\partial}{\partial z}$ that is used to find a particular solution of (5.1) which is the main target of this chapter. As we justified the recovery of (5.2) from (5.1), we called (5.1) a generalization of (5.2). Finally, we gave the solution of the homogeneous part of (5.1) where we followed the Frobenius method to have it given as in (5.39).

Appendix A

Important Results

In this Appendix we provide some definitions, theorems and detailed proofs of some of the results used in the main body of the Thesis.

Theorem A.0.1. *[31, Th 13] (*Weierstrass M-test*) Suppose*

$$\sum_{n=0}^{\infty} M_n$$

is a convergent series with real nonnegative terms and suppose, for all z in some set A and for all n greater than some number N, that

$$|f_n(x)| \leq M_n \, .$$

Then, the series

$$\sum_{n=0}^{\infty} f_n(z)$$

converges uniformly on A.

Definition A.0.2. *[26, (1.2)] We say that $f(z)$ is* asymptotically equivalent *or* equal *to $g(z)$*

under the limit $z \to z_0$ if f and g are such that $\lim_{z\to z_0} \frac{f(z)}{g(z)} = 1$. *We write*

$$f(z) \sim g(z) \ \textit{as}\ z \to z_0 \ \textit{if} \ \lim_{z\to z_0} \frac{f(z)}{g(z)} = 1. \tag{A.1}$$

For example, if $f(z) = z^2 + z\log z$, then $f(z) \sim z^2$ as $z \to \infty$. Also, note that (A.1) implies that $\lim_{z\to\infty} f(z) = \lim_{z\to\infty} g(z)$.

Theorem A.0.3. *[11, Eq.(5.78)]*

$$\beta(p,q;b) \leq exp(-4b)\beta(p,q) \quad (p>0, q>0, b\geq 0). \tag{A.2}$$

Here we generalize this result to the complex case in the following theorem.

Theorem A.0.4.

$$|\beta(x,y,;p)| \leq exp(-4p)|\beta(x,y)| \quad (x,y \in \mathscr{C}\, and\, p>0). \tag{A.3}$$

Proof. We follow same procedure of proof of (A.2) stated in [11, p224]. We start with the integral representation of the extended beta function give by (1.15):

$$\beta(x,y;p) = \int_0^1 t^{x-1}(1-t)^{y-1} exp\left[\frac{-p}{t(1-t)}\right] dt.$$

If we use the transformation $t = u/(u+1)$ in above integral, then it can be written as [11, eq (5.82)]

$$\beta(x,y;p) = exp(-2p)\int_0^\infty \frac{u^{x-1} exp\left[-p\left(u+u^{-1}\right)\right]}{(1+u)^{x+y}} du. \tag{A.4}$$

We note that the function $exp\left[-p\left(u+u^{-1}\right)\right]$ attains its maximum at $u=1$. This is because

$$f'(u) = -p\left(1-\frac{1}{u^2}\right) exp\left[-p\left(u+\frac{1}{u}\right)\right].$$

$$f''(u) = p^2\left(1-\frac{1}{u^2}\right) exp\left[-p\left(u+\frac{1}{u}\right)\right] - \frac{2p}{u^3} exp\left[-p\left(u+\frac{1}{u}\right)\right].$$

Setting $f'(u)=0$ gives $u=\mp u$. substituting in $f''(u)$ will till about the maximum and the minimum as in classical calculus.

$$f''(1) = -2pe^{-2p} < 0 \tag{A.5}$$

which indeed gives the maximum and

$$f''(-1) = 2pe^{-2p} > 0 \tag{A.6}$$

which gives the minimum.

By taking the absolute value of both sides of (A.4) we get

$$|\beta(x,y;p)| \leq exp\left(-4p\right)\left|\int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}\, du\right|. \tag{A.7}$$

The integral inside the absolute value in (A.7) is exactly the standard beta function written in its other integral form [2, eq (2.37)] and hence we obtain (A.3) as desired. □

As stated in the context that the proof of the following theorem is available in the literature [37]. However, here we provide our proof using our integral representation of the BF given by (3.19)

$$J_\alpha(z) = \frac{(z/2)^\alpha}{\Gamma(\alpha+1)}\left[1 - \frac{z}{2}\int_0^1 \frac{(1-t)^\alpha}{\sqrt{t}} J_1(z\sqrt{t})dt\right] \quad (Re(\alpha) > -1). \tag{A.8}$$

Theorem A.0.5.

$$\int_0^\infty e^{-at} J_0(ht)dt = \frac{1}{\sqrt{a^2+h^2}} \quad (Re(a) > 0). \tag{A.9}$$

Proof.

$$\begin{aligned}
&\int_0^\infty e^{-az} z^\alpha J_\alpha(bz)dz \\
&= \int_0^\infty e^{-az} z^\alpha \left[\frac{b^\alpha z^\alpha}{2^\alpha \Gamma(\alpha+1)} - \frac{b^{\alpha+1} z^{\alpha+1}}{2^{\alpha+1}} \int_0^1 \frac{(1-t)^\alpha}{\sqrt{t}} J_1(bz\sqrt{t})dt\right] dz \\
&= \int_0^\infty \frac{b^\alpha z^{2\alpha}}{2^\alpha \Gamma(\alpha+1)} e^{-az} dz - \int_0^\infty \int_0^1 \frac{b^{\alpha+1} z^{2\alpha+1} (1-t)^\alpha}{2^{\alpha+1}\Gamma(\alpha+1)\sqrt{t}} e^{-az} J_1(bz\sqrt{t})dtdz.
\end{aligned} \tag{A.10}$$

The first integral in (A.10) can be evaluated using the following result

$$\int_0^\infty e^{-az} z^{x-1} dz = a^{-x}\Gamma(x) \tag{A.11}$$

which can be easily proven by using the transformation ($u = az$) in the definition of the gamma function given by (1.1)

$$\Gamma(x) = \int_0^\infty e^{-z} z^{x-1} dz.$$

That is

$$\int_0^\infty e^{-az} z^{x-1} dz = \frac{1}{a}\int_0^\infty e^{-u} a^{1-x} u^{x-1} du = a^{-x}\Gamma(x) \tag{A.12}$$

Therefore, we have

$$\int_0^\infty \frac{b^\alpha z^{2\alpha}}{2^\alpha \Gamma(\alpha+1)} e^{-az} dz = \frac{b^\alpha}{2^\alpha \Gamma(\alpha+1)} a^{-2\alpha-1}\Gamma(2\alpha+1). \tag{A.13}$$

Substituting (A.13) in (A.10) gives

$$\begin{aligned}\int_0^\infty e^{-az} z^\alpha J_\alpha(bz)dz =& \frac{b^\alpha}{2^\alpha \Gamma(\alpha+1)} a^{-2\alpha-1}\Gamma(2\alpha+1)\\ &- \int_0^1 \frac{b^{\alpha+1}(1-t)^\alpha}{2^{\alpha+1}\Gamma(\alpha+1)\sqrt{t}} \left(\int_0^\infty e^{-az} z^{2\alpha+1} J_1(bz\sqrt{t})dz\right) dt. \quad \text{(A.14)}\end{aligned}$$

Now, by setting $\alpha = 0$ in (A.14) we get

$$\int_0^\infty e^{-az} J_0(bz)dz = \frac{1}{a} - \int_0^1 \frac{b}{2\sqrt{t}} \left(\int_0^\infty z e^{-az} J_1(bz\sqrt{t})dz\right) dt. \quad \text{(A.15)}$$

We can evaluate the integral inside the parentheses by using the following result [2, Eq 6.47]

$$\int_0^\infty e^{-az} z^p J_p(bz)dz = \frac{(2b)^p \Gamma(p+\frac{1}{2})}{\sqrt{\pi}(a^2+b^2t)^{p+\frac{1}{2}}} \quad (p > \frac{1}{2}, a > 0, b > 0). \quad \text{(A.16)}$$

So,

$$\int_0^\infty z e^{-az} J_1(bz\sqrt{t})dz = \frac{2b\sqrt{t}\Gamma(3/2)}{\sqrt{\pi}(a^2+b^2t)^{3/2}} = \frac{b\sqrt{t}}{\sqrt{\pi}(a^2+b^2t)^{3/2}} \quad \text{(A.17)}$$

since by [2, Eq 2.25], (the special case of Legendre duplication formula), $\Gamma(3/2) = \frac{\sqrt{\pi}}{2}$. Now, putting (A.17) in (A.15) gives

$$\int_0^\infty e^{-az} J_0(bz)dz = \frac{1}{a} - \int_0^1 \frac{b^2}{2(a^2+b^2t)^{3/2}} dt. \quad \text{(A.18)}$$

By using the transformation ($u = a^2 + b^2 t$) to evaluate the integral on the right side of (A.18), we have

$$\int_0^1 \frac{b^2}{2(a^2+b^2t)^{3/2}} dt = \int_{a^2}^{a^2+b^2} \frac{1}{2u^{3/2}} du = \frac{1}{a} - \frac{1}{\sqrt{a^2+b^2}}. \quad \text{(A.19)}$$

Substitution of (A.19) in (A.18) proves the theorem. □

In the following corollary, we show how to recover (4.24) from (4.25) when setting $p=0$.

Corollary A.0.6. *Setting $p=0$ in (4.25) given by*

$$\int_0^\infty e^{-az}z^{\mu-1}J_{\alpha,p}(hz)dz = \frac{h^\alpha\Gamma(\mu+\alpha)}{a^{\alpha+\mu}2^\alpha\Gamma(\alpha+1)}$$

$$\times\left[1-\frac{h^2(\mu+\alpha+1)(\mu+\alpha)}{4a^2(\alpha+1)}F_p\left(\begin{array}{cc}\frac{\mu+\alpha+2}{2} & \frac{\mu+\alpha+3}{2}\\ & 2\end{array};1;\alpha+2;-\frac{h^2}{a^2}\right)\right].$$

gives (4.24) which is

$$\int_0^\infty e^{-az}z^{\mu-1}J_\alpha(hz)dz$$

$$=\frac{\left(\frac{1}{2}h/a\right)^\alpha\Gamma(\mu+\alpha)}{a^\mu\Gamma(\alpha+1)}\,{}_2F_1\left(\frac{\mu+\alpha}{2};\frac{\mu+\alpha+1}{2};\alpha+1;-\frac{h^2}{a^2}\right)(|h|<|\alpha|).$$

Proof. Setting $p=0$ in (4.25) and using the series representation given by (4.31) we have

$$\int_0^\infty e^{-az}z^{\mu-1}J_{\alpha,0}(hz)dz$$

$$=\frac{h^\alpha a^{-\alpha-\mu}\Gamma(\alpha+\mu)}{2^\alpha\Gamma(\alpha+1)}-\sum_{n=0}^\infty\left(\frac{[(\alpha+\mu+2)/2]_n[(\alpha+\mu+3)/2]_n}{[2]_n}\right)$$
$$\times\left(\frac{(-1)^n h^{2n+\alpha+2}\Gamma(\alpha+\mu+2)\beta(n+1,\alpha+1)}{2^{\alpha+2}a^{2n+\alpha+\mu+2}\Gamma(\alpha+1)n!}\right). \tag{A.20}$$

By (1.7),

$$\beta(n+1,\alpha+1)=\frac{\Gamma(n+1)\Gamma(\alpha+1)}{\Gamma(n+\alpha+2)} \tag{A.21}$$

and so (A.20) becomes

$$\int_0^\infty e^{-az} z^{\mu-1} J_{\alpha,0}(hz)dz$$

$$= \frac{h^\alpha a^{-\alpha-\mu}\Gamma(\alpha+\mu)}{2^\alpha\Gamma(\alpha+1)} - \sum_{n=0}^\infty \left(\frac{[(\alpha+\mu+2)/2]_n[(\alpha+\mu+3)/2]_n}{[2]_n} \right)$$

$$\times \left(\frac{(-1)^n h^{2n+\alpha+2}\Gamma(\alpha+\mu+2)\Gamma(n+1)\Gamma(\alpha+1)}{2^{\alpha+2}a^{2n+\alpha+\mu+2}\Gamma(\alpha+1)n!\Gamma(n+\alpha+2)} \right). \tag{A.22}$$

Now, by (4.2), we have

$$(a+1)_n = \frac{a_{n+1}}{a}$$

and so,

$$[(\alpha+\mu+2)/2]_n = \left[\frac{\alpha+\mu}{2}+1\right]_n = \frac{[(\alpha+\mu)/2]_{n+1}}{(\alpha+\mu)/2}, \tag{A.23}$$

$$[(\alpha+\mu+3)/2]_n = \left[\frac{\alpha+\mu+1}{2}+1\right]_n = \frac{[(\alpha+\mu+1)/2]_{n+1}}{(\alpha+\mu+1)/2}. \tag{A.24}$$

Also, by iteration of (1.3), we have

$$\begin{aligned}\Gamma(n+\alpha+2) &= (n+\alpha+1)\Gamma(n+\alpha+1)\\ &= (n+\alpha+1)(n+\alpha)\Gamma(n+\alpha) = \cdots\\ &= (n+\alpha+1)(n+\alpha)\cdots(\alpha+1)\Gamma(\alpha+1)\\ &= (\alpha+1)_{n+1}\Gamma(\alpha+1).\end{aligned} \tag{A.25}$$

Therefore, as $(2)_n = (n+1)!$, then by (A.23),(A.24) and (A.25), the series on the right side of (A.22) becomes

$$\sum_{n=0}^\infty \left\{ \frac{4[(\alpha+\mu)/2]_{n+1}[(\alpha+\mu+1)/2]_{n+1}}{(\alpha+\mu)(\alpha+\mu+1)(n+1)!} \right\}$$

$$\times \left\{ \frac{(-1)^n h^{2n+\alpha+2}(\alpha+\mu+1)(\alpha+\mu)\Gamma(\alpha+\mu)n!\Gamma(\alpha+1)}{2^{\alpha+2}a^{2n+\alpha+\mu+2}\Gamma(\alpha+1)n!(\alpha+1)_{n+1}\Gamma(\alpha+1)} \right\}. \tag{A.26}$$

Cancelling common terms from (A.26) gives

$$\sum_{n=0}^{\infty}\frac{(-1)^n[(\alpha+\mu)/2]_{n+1}[(\alpha+\mu+1)/2]_{n+1}h^{2n+\alpha+2}\Gamma(\alpha+\mu)}{(n+1)!2^{\alpha}a^{2n+\alpha+\mu+2}\Gamma(\alpha+1)(\alpha+1)_{n+1}}. \tag{A.27}$$

By shifting the index in (A.27) one step backward by setting $n+1\to n$ we obtain

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{(-1)^{n-1}[(\alpha+\mu)/2]_n[(\alpha+\mu+1)/2]_n h^{2n+\alpha}\Gamma(\alpha+\mu)}{n!\,2^{\alpha}a^{2n+\alpha+\mu}\Gamma(\alpha+1)(\alpha+1)_n}\\ &=-\sum_{n=1}^{\infty}\frac{(-1)^{n}[(\alpha+\mu)/2]_n[(\alpha+\mu+1)/2]_n h^{2n+\alpha}\Gamma(\alpha+\mu)}{n!\,2^{\alpha}a^{2n+\alpha+\mu}\Gamma(\alpha+1)(\alpha+1)_n}.\end{aligned} \tag{A.28}$$

Now, by substituting (A.28) in (A.22) and taking $\frac{h^{\alpha}a^{-\alpha-\mu}\Gamma(\alpha+\mu)}{2^{\alpha}\Gamma(\alpha+1)}$ as common factor we get

$$\begin{aligned}\int_0^{\infty}e^{-az}z^{\mu-1}J_{\alpha,0}(hz)dz=&\frac{h^{\alpha}a^{-\alpha-\mu}\Gamma(\alpha+\mu)}{2^{\alpha}\Gamma(\alpha+1)}\\ &\times\left\{1+\sum_{n=1}^{\infty}\frac{[(\alpha+\mu)/2]_n[(\alpha+\mu+1)/2]_n(-h^2/a^2)^n}{n!(\alpha+1)_n}\right\}.\end{aligned} \tag{A.29}$$

Indeed, as $(a)_0=0$, (A.29) could be written as

$$\begin{aligned}&\int_0^{\infty}e^{-az}z^{\mu-1}J_{\alpha,0}(hz)dz\\ &=\frac{\left(\frac{1}{2}h/a\right)^{\alpha}\Gamma(\mu+\alpha)}{a^{\mu}\Gamma(\alpha+1)}\sum_{n=0}^{\infty}\frac{[(\alpha+\mu)/2]_n[(\alpha+\mu+1)/2]_n(-h^2/a^2)^n}{n!(\alpha+1)_n}.\end{aligned} \tag{A.30}$$

By noting that the series on the right side of (A.30) is nothing but

$${}_2F_1\left(\frac{\mu+\alpha}{2};\frac{\mu+\alpha+1}{2};\alpha+1;-\frac{h^2}{a^2}\right),$$

then (4.24) is obtained and hence the proof is complete. □

Definition A.0.7. *The integral exponential function is defined as [28]*

$$E_\nu = z^{\nu-1}\int_z^\infty t^{-\nu} exp\left(-t\right) dt. \tag{A.31}$$

Lemma A.0.8.

$$\int_0^z \tau^n \, exp\left(-c/\tau\right) d\tau = z^{n+1} E_{n+2}\left(\frac{c}{z}\right). \tag{A.32}$$

Proof. Use the transformation $u = c/\tau$ to have

$$\begin{aligned}
\int_0^z \tau^n \, exp\left(-c/\tau\right) d\tau &= c^{n+1}\int_{c/z}^\infty u^{-(n+2)} \, exp\left(-u\right) du \\
&= c^{n+1}\left(\frac{c}{z}\right)^{-(n+1)}\left(\frac{c}{z}\right)^{(n+1)}\int_{c/z}^\infty u^{-(n+2)} \, exp\left(-u\right) du \\
&= c^{n+1}\left(\frac{z}{c}\right)^{(n+1)}\left[\left(\frac{c}{z}\right)^{(n+1)}\int_{c/z}^\infty u^{-(n+2)} \, exp\left(-u\right) du\right].
\end{aligned} \tag{A.33}$$

By (A.31), the integral inside the bracket in (A.33) is $E_{n+2}\left(\frac{c}{z}\right)$. So, we have

$$\begin{aligned}
\int_0^z \tau^n \, exp\left(-c/\tau\right) d\tau &= c^{n+1}\left(\frac{z}{c}\right)^{(n+1)} E_{n+2}\left(\frac{c}{z}\right) \\
&= z^{n+1} E_{n+2}\left(\frac{c}{z}\right)
\end{aligned}$$

as desired. □

Theorem A.0.9. (Cauchy's Residue Theorem) *[31, Th 6.2]*

If C is a simple closed positively oriented contour and f is analytic inside and on C except at the points $z_1, z_2, \ldots, z_n$ inside C, then

$$\oint_C f(z) dz = 2\pi i \sum_{j=1}^{n} Res(z_j). \tag{A.34}$$

Lemma A.0.10. *The residue of* $\Gamma(z)$ *at* $z = -m$ *is*

$$\frac{(-1)^m}{m!} \tag{A.35}$$

where m is an integer and $m \geq 0$.

Proof. We use the following representation of the gamma function developed in (A.25)

$$\Gamma(z) = \frac{\Gamma(z+m)}{z(z+1)(z+2)\cdots(z+m-1)}.$$

Now, as the poles are simple poles at $\{0, -1, -2, ...\}$, we can find the residue at any of them by finding the limit $\lim_{z\to -m}(z+m)\Gamma(z)$. To compute this, we have

$$\lim_{z\to -m}(z+m)\Gamma(z+m) = \lim_{z\to -m}\frac{(z+m)\Gamma(z+m)}{z(z+1)(z+2)\cdots(z+m-1)}. \tag{A.36}$$

Using the relation $(z+m)\Gamma(z+m) = \Gamma(z+m+1)$ given by (1.3), then no more singularities would be present in the above limit. Therefore, we can be evaluated directly by substitution. So, (A.36) becomes

$$\begin{aligned}
\lim_{z\to -m}(z+m)\Gamma(z) &= \lim_{z\to -m}\frac{\Gamma(z+m+1)}{z(z+1)(z+2)\cdots(z+m-1)} \\
&= \frac{\Gamma(-m+m+1)}{(-m)(-m+1)(-m+2)\cdots(-m+m-1)} \\
&= \frac{\Gamma(1)}{(-m)(-m+1)(-m+2)\cdots(-1)} \\
&= \frac{(-1)^m}{m!}
\end{aligned}$$

as desired. □

Appendix B

Mellin-Barnes Integrals Method

It is known in asymptotic studies that their are many ways to determine the asymptotic behavior of a given function. Examples include . However, the Mellin-Barnes integral method is yet another way that uses the power of evaluting the integral under consideration as we will see.

It is to remark here that the method we now present is not particularized to the loop that used here but we mentioned it so because it is the type of loops we used in our work to find the Mellin-Barnes integral for the EBF. For more details of the method could be found in [28] where the full view is presented.

The integrals which are considered as a class of frequently occurring Mellin-Barnes integrals have the form

$$f(z) = \frac{1}{2\pi i}\int_C g(s)z^s ds, \tag{B.1}$$

where the contour C is a loop in the complex s plane beginning at $+\infty$, travelling in a clockwise sense and returning to $+\infty$ with suitable indentations to avoid poles. The integrand

$g(s)$ in (B.1) is assumed to have the form

$$g(s) = \frac{\prod_{r=1}^{m}\Gamma(b_r - \beta_r s)\prod_{r=1}^{n}\Gamma(1+\alpha_r s - a_r)}{\prod_{r=m+1}^{q}\Gamma(1+\beta_r s - b_r)\prod_{r=n+1}^{p}\Gamma(a_r - \alpha_r s)}$$

$$(0 \leq m \leq q,\ 0 \leq n \leq p). \tag{B.2}$$

The parameters α_r, β_r are restricted to be positive numbers and it is assumed that the parameters are such that the path C may be chosen to separate the sequences of poles resulting from $\Gamma(b_r - \beta_r s)\,(1 \leq r \leq m)$ from those of $\Gamma(1+\alpha_r s - a_r)\,(1 \leq r \leq n)$. The coefficients α_r, β_r of the integration variable in the gamma functions in (B.2) are termed the multiplicities of the associated gamma functions. Now we state the following lemma which is known as Rule 2 that provide a convergence criterion for the integral (B.1).

Lemma B.0.11 (Rule 2)**.** *The Mellin-Barnes integral (B.1) taken along the loop C, beginning at $+\infty$ and encircling in the negative sense only the poles of the integrand located at $\beta_r s = b_r + k$, $(1 \leq r \leq m)$ and k a nonnegative integer, converges for all finite, nonzero values of z provided*

$$\left\{\begin{array}{ccc} \textit{number of gamma functions} & & \textit{number of gamma functions} \\ \textit{with negative multiplicity in} & & \textit{with positive multiplicity in} \\ \textit{the numerator and with} & - & \textit{the numerator and with} \\ \textit{positive multiplicity in} & & \textit{negative multiplicity in} \\ \textit{the denominator} & & \textit{the denominator} \end{array}\right\} > 0,$$

each gamma function being counted according to its multiplicity.

The method of determination of the asymptotic expansion of $f(z)$ for large $|z|$ from (B.1) is a well-known and powerful technique. Suitable displacement of the path over a subset of the poles of $g(s)$ then produces expansions in either ascending or descending powers of the variable z. The expansion in descending powers corresponds to the asymptotic expansion of $f(z)$ valid as $|z| \to \infty$. The form of this expansion must, by the nature of the integral (B.1), be of algebraic type. That is the controlling behavior in the expansions is either an algebraic power of the variable z or contains an exponential functions of z.

Another powerful feature of this approach is that the remainder term which results when the path is displaced over a finite number of poles is simply given by the integral (B.1) taken over the displaced path. It is then usually a relatively straightforward matter to obtain a bound for the remainder to establish the nature of the expansion.

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